Chapter 1

MINIMIZATION AND MOUNTAIN-PASS THEOREMS

In this introductory chapter, we consider the concept on differentiability of mappings in Banach spaces, Fréchet and Gâteaux derivatives, second-order derivatives and general minimization theorems. Variational principles of Ekeland [Ek1] and Borwein & Preiss [BP] are proved and relations to the minimization problem are given. Deformation lemmata, Palais–Smale conditions and mountain-pass theorems are considered. The deformation approach and $\varepsilon$–variational approach are applied to prove the mountain-pass theorem and its various extensions.

Let $E$ be a Banach space, $X \subset E$ be an open subset, $f : X \rightarrow \mathbb{R}$ be a differentiable functional. Minimax theorems characterize a critical value of a functional $f$ as a minmax over a suitable class $\mathcal{A}$ of subsets of $X$

$$c = \inf_{A \in \mathcal{A}} \sup_{x \in A} f(x).$$

The basic ideas on minimax characterization of critical points go back to the work of Lusternik & Schnirelman [LS]. In this chapter, we present the mountain-pass theorem of Ambrosetti–Rabinowitz and some of its extensions. Their statements involve a compactness assumption, the so called Palais–Smale ($PS$) condition and variants. In a general form, it has been given by Palais [Pal]. During the last two decades, minimax theorems have
been extensively developed, generalizing the assumptions on differentiability of the functional, the \((PS)\) type conditions and geometric conditions on the functional. In a general form, the mountain-pass theorem has been proved for continuous functionals by Degiovanni & Marzocchi [DM]. For discontinuous functionals it has been proved by Ribarska, Tsachev and Krastanov [RTK1].

In this chapter we consider deformation theorems of Rabinowitz [Ra1], [Ra2], Willem [Wil1], [Wil3], Bartolo, Benci and Fortunato [BBF] and Palais–Smale type conditions of Cerami [Ce] \((PSC)_c\), Schechter [Sch1], [Sch2] \((PS)_{c,\psi}\), and Struwe [St1], [St2], \((PS)^\ast_c\)-condition in scales of Banach spaces. We introduce \((PS)_{c,\varphi}\)-condition which “lies between” \((PSC)_c\) and \((PS)_{c,\psi}\), and extend some deformation theorems. We prove the mountain-pass theorem of Ambrosetti & Rabinowitz [Ara] and its extensions due to Cerami [Ce], Willem [Wil1], Pucci & Serin [PS1], Rabinowitz [Ra1], Schechter [Sch1], [Sch2], Brézis & Nirenberg [BN], Aubin & Ekeland [AE], Ghossoub & Preiss [GP]. A variant of a three critical point theorem with \((PS)_{c,\psi}\) condition is proved.

1.1 Differential Calculus for Mappings in Banach Spaces

1.1.1 Fréchet and Gâteaux Derivatives

A natural extension of the derivative of a function of one variable is the Fréchet derivative of a mapping in a Banach space.

Let \(X\) and \(Y\) be Banach spaces with norms \(||\cdot||_X\) and \(||\cdot||_Y\) respectively. Let \(U \subset X\) be an open subset and \(F : U \to Y\) be a mapping. When \(Y = \mathbb{R}\), \(F\) is said to be a functional.

We use the notation \(r(h) = o(||h||_X)\) for the mapping \(r : X \to Y\) if and only if (iff)

\[
\lim_{h \to 0} \frac{||r(h)||_Y}{||h||_X} = 0,
\]

which means that for every \(\varepsilon > 0\) there exists \(\delta > 0\) such that if \(||h||_X < \delta\) then \(||r(h)||_Y < \varepsilon ||h||_X\).

**Definition 1.1.** Let \(x\) be a point of the open subset \(U \subset X\). The mapping \(F : U \to Y\) is Fréchet-differentiable at \(x \in U\) if there exists a linear operator \(A \in L(X,Y)\) such that

\[
F(x + h) - F(x) - Ah = o(||h||). 
\] (1.1)
The operator $A$ is said to be Fréchet derivative of the mapping $F$ at $x$ and can be denoted as $DF(x)$ or $F'(x)$. It has the linear property

$$A(c_1 h_1 + c_2 h_2) = c_1 Ah_1 + c_2 Ah_2,$$

for all $h_1, h_2 \in X$ and $c_1, c_2 \in \mathbb{R}$. Let $F : U \to Y$ be differentiable at every point of $U$. The mapping $DF : U \to L(X, Y)$ is called the Fréchet-derivative of $F$. We say that $F$ is a $C^1$ mapping iff $DF$ is continuous as a mapping from $U$ to $L(X, Y)$.

Next we present some other properties of the Fréchet derivative.

1. The operator $A = DF(x)$ satisfying (1.1) is unique.
2. If $F : U \to Y$ is Fréchet-differentiable at $x \in U$, then $F$ is continuous at $x$.
3. If $F : U \to Y$ is Fréchet-differentiable according to a norm in $X$, then it is Fréchet-differentiable according to any norm equivalent to the first norm.
4. If $F, G : U \to Y$ are Fréchet-differentiable at $x \in U$, then $aF + bG$, where $a, b \in \mathbb{R}$, is Fréchet-differentiable at $x \in U$ and

$$D(aF + bG)(x)h = aDF(x)h + bDG(x)h.$$

5. Let $F : U \to Y$, $G : V \to Z$ be mappings with $U$ and $V$ open subsets of $X$ and $Y$ such that $V \supset F(U)$. Let $G \circ F : U \to Z$ be the composition mapping. If $F$ is Fréchet-differentiable at $x \in U$ and $G : V \to Z$ is Fréchet-differentiable at $y = F(x) \in V$, then $GF$ is Fréchet-differentiable at $x$ and

$$D(G \circ F)(x)h = DG(y)DF(x)h.$$

Denote $\mathbb{R}^+ = [0, +\infty)$.

We give some examples of Fréchet derivatives.

Let $H$ be a Hilbert space with scalar product $(.,.)$ and norm $||.||$.

1. The functional $F : H \to \mathbb{R}^+$

$$F(x) = \frac{1}{2} ||x||^2 = \frac{1}{2} (x, x)$$

is Fréchet-differentiable and

$$F'(x)h = (x, h).$$

2. The functional $F : H \to \mathbb{R}^+$

$$F(x) = ||x||$$
is Fréchet-differentiable for \( x \neq 0 \) and
\[
F'(x) h = \frac{(x, h)}{\|x\|}, \quad x \neq 0.
\]

This functional is not Fréchet-differentiable at 0.

3. The functional \( F(x) = \frac{1}{2}(Ax, x) + (b, x) \), where \( A \in L(H, H) \) and \( b \in H \), is Fréchet-differentiable and
\[
F'(x) h = (Ax + b, h).\]

4. Let \( X = \mathbb{R}^n, Y = \mathbb{R}^m, x = (x_1, \ldots, x_n) \) and \( f \in C^1(\mathbb{R}^n, \mathbb{R}^m) \) be the mapping \( f(x) = [f_1(x), \ldots, f_m(x)]^T \), where \( B^T \) means the transpose matrix of the matrix \( B \).

Then \( A = f'(x) \in L(\mathbb{R}^n, \mathbb{R}^m) \) and
\[
A = f'(x) = \begin{bmatrix}
\frac{\partial f_1}{\partial x_1}(x) & \cdots & \frac{\partial f_1}{\partial x_n}(x) \\
\vdots & & \vdots \\
\frac{\partial f_m}{\partial x_1}(x) & \cdots & \frac{\partial f_m}{\partial x_n}(x)
\end{bmatrix}.
\]

For a differentiable functional \( f : X \to \mathbb{R} \) we have \( f'(x) \in L(X, \mathbb{R}) = X^* \), where \( X^* \) is the dual space of \( X \). Since it will be clear from the context we keep \( ||\cdot|| \) to denote also the norm in \( X^* \).

Let \( X = H \) be a Hilbert space with inner product \((.,.)\). By the Riesz representation theorem there exists a unique element \( \nabla f(x) \in H \) such that
\[
f'(x) h = (\nabla f(x), h), \quad \forall h \in H.
\]

The operator \( \nabla f : H \to H \) is called a potential operator with the potential \( f : H \to \mathbb{R} \).

Many equations of Mathematical Physics have the operator form \( f'(x) = 0 \) in an appropriate Hilbert space \( H \). The equation \( f'(x) = 0 \) is said to be the Euler–Lagrange equation of the functional \( f : H \to \mathbb{R} \). Its solutions are assumed in the weak sense, i.e.,
\[
(\nabla f(x), h) = 0, \quad \forall h \in H,
\]
and are considered as critical points of the functional \( f : H \to \mathbb{R} \).

Another derivative of a functional \( f \) is the directional derivative or so called Gâteaux derivative of \( f \).

**Definition 1.2.** Let \( F : U \to Y \) be a mapping and \( x \in U \). We say that \( F \) is Gâteaux-differentiable at \( x \) if there exists \( A \in L(X, Y) \), such that
\[
\lim_{t \to 0} \frac{||F(x + th) - F(x)||_Y}{t} = Ah, \quad \forall h \in X. \tag{1.2}
\]
The mapping $A$ is uniquely determined. It is called \textit{Gâteaux derivative of $f$ at $x$} and is denoted by $D_G F(x)$ or $F'_G(x)$. If $F$ is Fréchet-differentiable it is clear that it is Gâteaux-differentiable. The converse is not true which can be seen in the following example in $\mathbb{R}^2$.

\textbf{Example 1.1.} The function $f : \mathbb{R}^2 \to \mathbb{R}$

\[
f(x, y) = \begin{cases} 
  \left(\frac{x^2 y}{x^4 + y^2}\right)^2, & y \neq 0, \\
  0, & y = 0,
\end{cases}
\]

is Gâteaux-differentiable at $(0,0)$, but not Fréchet-differentiable at $(0,0)$.

\textit{Proof.} It is easy to show that $f$ is Gâteaux-differentiable at $(0,0)$. If $h = (h_1, h_2)$, $h_2 \neq 0$ we have

\[
\lim_{t \to 0} \frac{f(th) - f(0)}{t} = \lim_{t \to 0} \frac{t (h_2^2 h_2)^2}{(t^2 h_1^4 + h_2^4)^2} = 0.
\]

If $f$ is Fréchet-differentiable at $(0,0)$ it should be $f'(0,0) = 0$. This is not true because taking $h = (h_1, h_1^2) \to (0,0)$ we get

\[
\lim_{\|h\| \to 0} \frac{|f(h) - f(0)|}{\|h\|} = \lim_{h_1 \to 0} \left( \frac{h_1^4}{h_1^4 + h_1^4} \right)^2 \frac{1}{\sqrt{h_1^2 + h_1^4}} = \frac{1}{4} \lim_{h_1 \to 0} \frac{1}{\sqrt{h_1^2 + h_1^4}} = \infty.
\]

\textit{EndProof}

Note that the Gâteaux-differentiability at a point of a mapping does not imply even the continuity at this point. For example the function

\[
g(x, y) = \begin{cases} 
  1 & \text{if } y = x^2 \\
  0 & \text{if } y \neq x^2
\end{cases}
\]

is Gâteaux differentiable at $(0,0)$, but is not continuous at $(0,0)$.

Although there is a result when Gâteaux-differentiability imply Fréchet-differentiability.

Denote by $\langle , \rangle$ the duality pairing between $X^*$ and $X$ and shortly $\lim_j$ instead of $\lim_{j \to \infty}$. We say that $f$ is a $C^1$ functional, and write $f \in C^1(U, \mathbb{R})$, 
if the Fréchet-derivative $f'(x)$ exists at every point $x$ of $U$ and the mapping $x \mapsto f'(x)$ is continuous from $U$ to $X^*$, i.e., if $\lim_j x_j = x \in U$ then

$$\lim_j \langle f'(x_j) - f'(x), v \rangle = 0, \quad \text{uniformly on } \{v \in X : \|v\| \leq 1\}.$$ 

**Theorem 1.1.** Suppose that $f : U \to \mathbb{R}$ has a continuous Gâteaux derivative on $U$. Then $f$ is Fréchet-differentiable and $f \in C^1(U, \mathbb{R})$.

This result follows by a variant of the mean-value theorem.

**Theorem 1.2.** Let $f : U \to \mathbb{R}$ be a Gâteaux-differentiable on $U$ and $x_1, x_2 \in U$. Then

$$|f(x_1) - f(x_2)| \leq \sup_{t \in [0, 1]} \|D_G f(x_1 + t(x_2 - x_1))\| \cdot \|x_1 - x_2\|$$

Let $\Omega$ be an open subset of $\mathbb{R}^N$ with finite measure $|\Omega| < \infty$. Denote by $L^q(\Omega), 1 < q < \infty$ the usual Lebesgue space of integrable functions.

**Example 1.2.** The functional $\varphi : L^{p+1}(\Omega) \to \mathbb{R}$, $1 < p < \infty$,

$$\varphi(u) = \frac{1}{p+1} \int_{\Omega} |u(x)|^{p+1} dx,$$

is of class $C^1(L^{p+1}(\Omega), \mathbb{R})$ and

$$\langle \varphi'(u), h \rangle = \int_{\Omega} u(x) |u(x)|^{p-1} h(x) dx.$$ 

**Proof.** We use Theorem 1.1 and show that there exists $\varphi'_G$ and that it is continuous. Let $u, h \in L^{p+1}(\Omega)$ and $t \in [0, 1]$. By the mean-value theorem, there exists $\xi \in [0, 1]$ such that

$$\frac{1}{(p + 1)|t|} |u(x) + th(x)|^{p+1} - |u(x)|^{p+1}|$$

$$= |u(x) + t\xi h(x)|^p |h(x)| \leq |u(x)| + |h(x)|^p |h(x)|.$$ 

By Hölder inequality, it follows
\[
\int_{\Omega} ||u(x)| + |h(x)||^p |h(x)| \, dx
\leq \left( \int_{\Omega} ||u(x)| + |h(x)||^{p+1} \, dx \right)^{\frac{p}{p+1}} \left( \int_{\Omega} |h(x)|^{p+1} \, dx \right)^{\frac{1}{p+1}}
\leq \left( 2^p \int_{\Omega} \left( |u(x)|^{p+1} + |h(x)|^{p+1} \right) \, dx \right)^{\frac{p}{p+1}} \left( \int_{\Omega} |h(x)|^{p+1} \, dx \right)^{\frac{1}{p+1}}
< \infty.
\]

Then, by Lebesgue theorem, it follows

\[
\langle \varphi'_G(u), h \rangle = \lim_{t \to 0} \frac{1}{(p+1) t} \int_{\Omega} |u(x) + th(x)|^{p+1} - |u(x)|^{p+1} \, dx
\]

\[
= \lim_{t \to 0} \int_{\Omega} |u(x) + t\xi h(x)|^p \text{sgn}(u(x) + t\xi h(x)) h(x) \, dx
\]

\[
= \int_{\Omega} |u(x)|^p \text{sgn}u(x) h(x) \, dx
\]

\[
= \int_{\Omega} u(x) |u(x)|^{p-1} h(x) \, dx.
\]

To prove the continuity of \( \varphi'_G(u) \) we need to prove that if \( \lim_j u_j = u \) in \( L^{p+1}(\Omega) \) then

\[
\lim_j \langle \varphi'_G(u_j) - \varphi'_G(u), v \rangle = 0, \quad \text{if } ||v||_{L^{p+1}} \leq 1. \tag{1.3}
\]

By the continuity of so-called superposition (or Nemitskii) operator (see Vainberg [V]) \( g : L^{p+1}(\Omega) \to L^{\frac{p+1}{p}}(\Omega) \)

\[
g(u) := |u|^{p-1},
\]

it follows that

\[
|\langle \varphi'_G(u_j) - \varphi'_G(u), v \rangle| \leq ||g(u_j) - g(u)||_{L^{\frac{p+1}{p}}} ||v||_{L^{p+1}} \to 0,
\]
which proves (1.4). End Proof

Recall the notion of superposition (or Nemitskii) operator. Let $\Omega$ be an open subset of $\mathbb{R}^N$ with finite measure $f \in C^0(\bar{\Omega} \times \mathbb{R})$ and $1 \leq p, q < \infty$. The operator

$$N_f u(x) := f(x, u(x)),$$

is called a superposition operator.

A function $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is called a Carathéodory function if:

1. for each fixed $s \in \mathbb{R}$, the function $x \mapsto f(x, s)$ is Lebesgue measurable in $\Omega$,
2. for almost every $x \in \Omega$, the function $s \mapsto f(x, s)$ is continuous on $\mathbb{R}$.

**Theorem 1.3.** Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function. Then:

1. The function $x \mapsto f(x, u(x))$ is a measurable function for every measurable function $u : \Omega \to \mathbb{R}$.
2. If $\Omega$ has finite measure, the Nemitskii operator $N_f : \mathcal{M} \to \mathcal{M}$ is a continuous, where $\mathcal{M}$ is the space of real-valued measurable functions on $\Omega$, equipped with the topology of convergence in measure.
3. If $\Omega$ is a bounded domain and $f$ satisfies the growth condition

$$|f(x, s)| \leq a|s|^{p-1} + b(x), \quad (1.4)$$

for $p > 1, a > 0$ and $b(x) \in L^q(\Omega)$, where $1/p + 1/q = 1$, then the Nemitskii operator $N_f : L^p(\Omega) \to L^q(\Omega)$ is continuous.
4. Let $N_F$ be the Nemitskii operator associated to the function

$$F(x, s) = \int_0^s f(x, t) dt,$$

where $f$ satisfies (1.4). Then $N_F : L^p(\Omega) \to L^1(\Omega)$ is a continuous operator. Moreover $\mathcal{F}(u) = \int_\Omega F(x, u(x)) dx$ defines a continuously Fréchet-differentiable functional and $\mathcal{F}'(u) = N_f$.

We refer the reader to Berger [B], Vainberg [V] for the properties of the superposition operator.

### 1.1.2 Second-order derivatives

Let $F : X \to Y$ be a differentiable mapping in the open set $U \subset X$ and consider $F : U \to L(X, Y)$. If this mapping is differentiable at a point
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If \( x \in U \), then its derivative \( (F')' (x) \in L(X, L(X,Y)) \) is said to be second derivative of \( F \) at \( x \) and can be noted as \( F'' (x) \) or \( D^2 F (x) \). It is convenient to consider \( D^2 F (x) \) as a bilinear map on \( X \). We recall that \( B : X \times X \to Y \) is a continuous bilinear map if

1. For every \( x = (x_1, x_2) \) and \( x' = (x'_1, x'_2) \in X \times X \) and every \( \alpha \) and \( \beta \in \mathbb{R} \)

\[
B(\alpha x_1 + \beta x_2, x'_1) = \alpha B(x_1, x'_1) + \beta B(x_2, x'_1),
\]

\[
B(x_1, \alpha x'_1 + \beta x'_2) = \alpha B(x_1, x'_1) + \beta B(x_1, x'_2).
\]

2. There exists a positive number \( M > 0 \) such that

\[
\|B(x, x')\|_Y \leq M \|x\|_X \|x'\|_X.
\]

The norm of bilinear map \( B \) is defined as

\[
\|B\|_b = \sup \left\{ \|B(x, x')\|_Y : \|x\|_X \leq 1, \|x'\|_X \leq 1 \right\}.
\]

The space of bilinear maps from \( X \times X \) to \( Y \) is denoted by \( B(X^2, Y) \). There is a natural isometry \( i : L(X, L(X,Y)) \to B(X^2, Y) \) defined by

\[
(iA)(x, x') = Ax(x'), \quad \forall (x, x') \in X^2, \|A\| = \|iA\|_b.
\]

Therefore we can consider \( F''(x) \) as an element of \( B(X^2, Y) \). The map \( F'' : U \to B(X^2, Y) \) is said to be the second Fréchet derivative of \( F \).

If \( F'' \) is continuous from \( U \) to \( B(X^2, Y) \) we say that \( F \in C^2(X,Y) \). If \( F \in C^2(X,Y) \) we have the Taylor’s formula

\[
F(x + h) = F(x) + F'(x)h + \frac{1}{2} F''(x)(h, h) + o(\|h\|^2_X).
\]

We recall that a linear operator \( L : X \to Y \) is called a Fredholm operator if the dimension of \( \mathcal{N}(L) \) and codimension of \( \mathcal{R}(L) \) are finite, where \( \mathcal{N}(L) \) and \( \mathcal{R}(L) \) denote the kernel and the range of \( L \) respectively. This implies that \( \mathcal{R}(L) \) is closed.

Let \( X = H \) be a Hilbert space, \( Y = \mathbb{R} \), \( U \subset H \) be an open subset and \( F \in C^2(U, \mathbb{R}) \). From the Riesz representation theorem there exists unique \( Lh \in H \) such that

\[
F''(x)(h,k) = (Lh, k),
\]
for all $k \in H$. The operator $L : H \to H$ is symmetric, i.e.,
\[(Lh,k) = (h,Lk), \quad \forall h, k \in H,
\]
and we identify $L = F''(x)$. If $F''(x)$ is a Fredholm operator then
\[H = \mathcal{N}(F''(x)) \oplus \mathcal{R}(F''(x)).\]

Assume that $x$ is a critical point of $F \in C^2(H, \mathbb{R})$. The point $x$ is said to be a non-degenerate critical point if $L = F''(x) : H \to H$ is an isomorphism. It is shown in Cartan [Car], that if $x$ is a non-degenerate critical point and $L = F''(x)$ is a positive-definite operator, that is,
\[(Lh,h) \geq 0, \quad \forall h \in H,
\]
then there exists a positive number $\lambda > 0$ such that
\[(Lh,h) \geq \lambda \|h\|^2. \quad (1.6)
\]

We use the notation $L_1 \geq L_2$ for symmetric operators $L_1$ and $L_2 : H \to H$ iff
\[(L_1 h,h) \geq (L_2 h,h), \quad \forall h \in H.
\]

If $x$ is a non-degenerate critical point and $L = F''(x)$ is a positive-definite operator, from Taylor’s formula (1.5), we have
\[
F(x + h) - F(x) = \frac{1}{2} (Lh,h) + o(\|h\|^2) \\
\geq \frac{\lambda}{2} \|h\|^2 + o(\|h\|^2).
\]

And it follows that $x$ is a point of strict local minima, see [Car].

In an analogous way, if $x$ is a non-degenerate critical point and $L$ is negative-definite, that is,
\[(Lh,h) \leq 0, \quad \forall h \in H,
\]
then $x$ is a point of strict local maxima.

If $L$ is an indefinite operator the critical saddle points are distinguished with respect to the so called Morse index. The Morse index of the critical point $x$ of the functional $F \in C^2(H, \mathbb{R})$ is defined as the supremum of the dimensions of the vector subspaces of $H$ on which $L = F''(x)$ is negative-definite. The nullity of $x$ is defined as the dimension of $\mathcal{N}(F''(x))$.

For detailed proofs of above mentioned statements we refer the reader to Ambrosetti & Prodi [APr], Cartan [Car], Kolmogorov & Fomin [KF].
1.2 Variational Principles and Minimization

1.2.1 Lower Semicontinuous Functions

Let $X$ be a Banach space $f : X \to \mathbb{R}$ a functional bounded from below. A sequence $(x_j)_j$ is said to be a minimizing sequence if

$$\lim_{j} f(x_j) = \inf_{x \in X} f(x).$$

The functional $f : X \to \mathbb{R}$ is said to be lower semi-continuous (respectively weakly lower semi-continuous) if whenever $\lim_j x_j = x$ strongly ($\lim_j x_j = x$ weakly), it follows

$$\liminf_{j \to \infty} f(x_j) \geq f(x).$$

The functional $f : X \to \mathbb{R}$ is sequentially weakly continuous if whenever $\lim_j x_j = x$ weakly, it follows

$$\lim_j f(x_j) \geq f(x).$$

Some properties of semi-continuity follow from the definition:

1. The sum of two l.s.c (w.l.s.c.) functionals is a l.s.c (w.l.s.c.) functional.

2. The product of l.s.c (w.l.s.c.) functionals with positive constant is a l.s.c (w.l.s.c.) functional.

3. If $(f_j)_j$ is a family of l.s.c (w.l.s.c.) functionals then the function $\sup_j f_j$ is a l.s.c (w.l.s.c.) functional.

We present a criterion for weak lower semicontinuity (see Berger [Ber], Chapter 6).

**Theorem 1.4.** Let $X$ be a reflexive Banach space, $f : X \to \mathbb{R}$ be a functional represented as the sum $f = f_1 + f_2$, where $f_1$ is continuous and convex and $f_2$ is sequentially weakly continuous. Then $f$ is weakly lower semi-continuous.

We have the following results for minimization (see Mawhin & Willem [MW2] and Berger [Ber]).

**Theorem 1.5.** Let $f$ be a weakly lower semi-continuous functional on the reflexive Banach space $X$ with a bounded minimizing sequence. Then $f$ has a minimum on $X$.

A functional $f$ is said to be coercive if $f(x) \to \infty$ as $\|x\| \to \infty$.

**Theorem 1.6.** Let $f$ be a weakly lower semi-continuous functional bounded from below on the reflexive Banach space $X$. If $f$ is coercive, then $c = \inf f$ is attained at a point $x_0 \in X$. 
1.2.2 Ekeland Theorem in Complete Metric Spaces

Let $M$ be a complete metric space and $\Phi : M \to \mathbb{R}$ a lower semi-continuous functional, bounded below. If $(u_j)_j$ is a minimizing sequence, then for every $\varepsilon > 0$ there exists $j_0$ such that for $j > j_0$

$$\Phi (u_j) \leq \inf_M \Phi + \varepsilon.$$

We say that $u$ is an $\varepsilon$–minimum point of $\Phi$ if

$$\Phi (u) \leq \inf_M \Phi + \varepsilon.$$

Ekeland theorem [Ek1] considers the existence of $\varepsilon$–minimum points.

**Theorem 1.7** (Ekeland Principle, strong form, 1979). Let $M$ be a complete metric space and $\Phi : M \to \mathbb{R}$ be a lower semicontinuous functional which is bounded from below. Let $k > 1$, $\varepsilon > 0$ and $u \in M$ be an $\varepsilon$–minimum point of $\Phi$. Then there exists $v \in M$ such that

$$\Phi (v) \leq \Phi (u), \quad (1.7)$$

$$d (u,v) \leq \frac{1}{k}, \quad (1.8)$$

$$\Phi (v) < \Phi (w) + \varepsilon kd (w,v), \quad \forall w \neq v. \quad (1.9)$$

**Proof.** Denote for simplicity $d_k (u,v) := kd (u,v)$ and define a partial ordering in $M$

$$u \prec v \iff \Phi (u) \leq \Phi (v) - \varepsilon d_k (u,v).$$

We have

$$u \prec u, \quad \forall u \in M,$$

$$u \prec v, \quad v \prec u \Rightarrow u = v, \quad \forall u, v \in M,$$

$$u \prec v, \quad v \prec w \Rightarrow u \prec w, \quad \forall u, v, w \in M.$$

Let us prove the transitivity. Assume that for $u, v, w \in M, u \prec v$ and $v \prec w$, which means

$$\Phi (u) \leq \Phi (v) - \varepsilon d_k (u,v),$$

and

$$\Phi (v) \leq \Phi (w) - \varepsilon d_k (v,w).$$
We prove that
\[ \Phi(u) \leq \Phi(w) - \varepsilon d_k(u, v). \]

Indeed we have
\[
\begin{align*}
\Phi(u) &\leq \Phi(v) - \varepsilon d_k(u, v) \\
&\leq \Phi(w) - \varepsilon (d_k(v, w) + d_k(u, v)) \\
&\leq \Phi(w) - \varepsilon d_k(u, w).
\end{align*}
\]

Now define a sequence of subsets \((S_n)_n\). Let \(u_1 = u\) and \(S_1 := \{w \in M : w \prec u_1\}\). We construct inductively a sequence \((u_n)_n\) as follows:

\[
\begin{align*}
u_2 &\in S_1, \quad \Phi(u_2) \leq \inf_{S_1} \Phi + \frac{\varepsilon}{2^n}, \\
S_2 &= \{w \in M : w \prec u_2\}, ... \\
u_{n+1} &\in S_n, \quad \Phi(u_{n+1}) \leq \inf_{S_n} \Phi + \frac{\varepsilon}{2^{n+1}}, \\
S_n &= \{w \in M : w \prec u_n\}.
\end{align*}
\]

We have
\[
S_1 \supset S_2 \supset ... \supset S_n \supset ...
\]
\[ u_1 \succ u_2 \succ ... \succ u_n \succ ... \]

Each \(S_n\) is closed. Indeed let \(v_j \in S_n\) and \(\lim_j v_j = v \in M\) which means
\[ \Phi(v_j) \leq \Phi(u_n) - \varepsilon d_k(v_j, u_n). \]

Letting \(j \to \infty\), by the lower semicontinuity of \(\Phi\) and continuity of the distance \(d_k\), we get
\[ \Phi(v) \leq \Phi(u_n) - \varepsilon d_k(v, u_n), \]

which means that \(v \in S_n\).

Next we have
\[ \lim_{n \to \infty} \text{diam} S_n = 0. \quad (1.10) \]

Indeed let \(w \in S_n\).
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$$\Phi (w) \leq \Phi (u_n) - \varepsilon d_k (w, u_n).$$

By $w \in S_n \subset S_{n-1}$

$$\Phi (u_n) \leq \inf_{S_{n-1}} \Phi + \frac{\varepsilon}{2n} \leq \Phi (w) + \frac{\varepsilon}{2n}$$

and

$$\Phi (u_n) - \frac{\varepsilon}{2n} \leq \Phi (w) \leq \Phi (u_n) - \varepsilon d_k (w, u_n).$$

So, it follows

$$d_k (w, u_n) \leq \frac{1}{2n}, \quad \forall w \in S_n.$$

Then for $w_1$ and $w_2 \in S_n$

$$d_k (w_1, w_2) \leq d_k (w_1, u_n) + d_k (w_2, u_n) \leq \frac{1}{2^{n-1}},$$

which proves (1.10).

From the principle of included intervals, there exists unique $v \in M$

$$\bigcap_{n=1}^{\infty} S_n = \{v\}.$$

We prove that $v$ satisfies (1.7)-(1.9). Since $v \in S_1$ and $v \prec u_1 = u$ it follows

$$\Phi (v) \leq \Phi (u) - \varepsilon d_k (u, v) \leq \Phi (u)$$

which is (1.7).

Let $w \neq v$. If $w \prec v$ it follows $w \in \bigcap_{n=1}^{\infty} S_n$ and then $w = v$. Therefore

$$\Phi (w) > \Phi (v) - \varepsilon d_k (w, v)$$

which is (1.9). Finally, by $\lim_n u_n = v$ and

$$d_k (u, u_n) \leq \sum_{j=1}^{n-1} d_k (u_j, u_{j+1}) \leq \sum_{j=1}^{n-1} \frac{1}{2^j} \leq 1,$$

it follows that $d_k (u, v) \leq 1$, which completes the proof. End Proof.

We present some corollaries derived from the Ekeland principle.

**Corollary 1.2** (Ekeland principle, weak form, 1979). Let $(M, d)$ be a complete metric space and $\Phi : M \to \mathbb{R}$ be a lower semicontinuous functional.
bounded from below. Then for every $\varepsilon > 0$ there exists an $\varepsilon$-minimum point of $\Phi$, $v \in M$ such that

$$\Phi (v) < \Phi (w) + \varepsilon d (w, v), \quad \forall w \in M, w \neq v.$$  

**Corollary 1.3.** Let $(M, d)$ be a complete metric space, $\Phi : M \to \mathbb{R}$ be lower semicontinuous functional bounded from below. Let $\varepsilon > 0$ and $u \in M$ be an $\varepsilon$-minimum point of $\Phi$. Then there exists $v \in M$ such that

$$\Phi (v) \leq \Phi (u),$$  
$$d (u, v) \leq \sqrt{\varepsilon},$$  
$$\Phi (v) < \Phi (w) + \sqrt{\varepsilon} d (w, v), \quad \forall w \neq v.$$

### 1.2.3 Palais–Smale Conditions and Minimization

Minimizing sequences for differentiable functionals are convergent under certain compactness conditions. We shall use later so called Palais–Smale ((PS) for short) conditions.

Let $X$ be a Banach space, $f : X \to \mathbb{R}$ be a differentiable functional.

**Definition 1.3** (Palais, 1970). A $C^1$-functional $f : X \to \mathbb{R}$ satisfies the Palais–Smale (PS) condition if every sequence $(x_j)_j$ in $X$ such that $f (x_j)$ is bounded and $\lim_j f' (x_j) = 0$ in $X^*$ has a convergent subsequence.

From (PS) condition, it follows that the set of critical points for a bounded functional is compact. A variant of (PS) condition, noted as (PS)$_c$, was introduced by Brézis, Coron and Nirenberg [BCN].

**Definition 1.4** (Brézis, Coron, Nirenberg, 1980). Let $c \in \mathbb{R}$. A $C^1$ functional $f : X \to \mathbb{R}$ satisfies the (PS)$_c$ condition if every sequence $(x_j)_j$ in $X$ such that $\lim_j f (x_j) = c$ and $\lim_j f' (x_j) = 0$ in $X^*$ has a convergent subsequence.

It is clear that (PS) condition implies the (PS)$_c$ condition for every $c \in \mathbb{R}$. The (PS)$_c$ condition implies the compactness of the set of critical points at a fixed level $c$.

**Theorem 1.8.** Let $f : X \to \mathbb{R}$ be a $C^1$ functional bounded below. Then, for each $\varepsilon > 0$ and $x \in X$ such that

$$f (x) \leq \inf_X f + \varepsilon,$$

there exists $y \in X$ such that

$$f (y) \leq f (x),$$  
$$\|x - y\| \leq \sqrt{\varepsilon},$$  
$$\|f' (y)\| \leq \sqrt{\varepsilon}.$$
Proof. By Corollary 1.2 applied to $M = X$ and $\Phi = f$ we have that there exists $y$ such that

$$f(z) > f(y) - \sqrt{\varepsilon}||y - z||, \quad \forall z \neq y. \quad (1.11)$$

Taking $z = y + th, \ t > 0, \ h \in X, \ ||h|| = 1$, in (1.11) we get

$$f(y + th) - f(y) > -\sqrt{\varepsilon}t.$$ 

Letting $t \to 0$ we obtain

$$\langle f'(y), h \rangle \geq -\sqrt{\varepsilon}, \quad \forall h \in X, \ ||h|| = 1.$$

Changing $h$ with $-h$ we have

$$-\sqrt{\varepsilon} \leq \langle f'(y), h \rangle \leq \sqrt{\varepsilon}, \quad \forall h \in X, \ ||h|| = 1,$$

which means that $||f'(y)|| \leq \sqrt{\varepsilon}$. End Proof

By Theorem 1.8 it follows

Corollary 1.3. Let $f : X \to \mathbb{R}$ be a $C^1$ functional bounded from below and $(x_j)_j$ be a minimizing sequence. Then there exists another minimizing sequence $(y_j)_j$ such that

$$f(y_j) \leq f(x_j),$$

$$\lim_j ||x_j - y_j|| = 0,$$

$$\lim_j ||f'(y_j)|| = 0.$$ 

Now, combining with $(PS)$ condition we get

Theorem 1.9. Let $f : X \to \mathbb{R}$ be a $C^1$-functional bounded below and $c = \inf f$. Assume that $f$ satisfies $(PS)_c$ condition. Then $c$ is achieved at a point $x_0 \in X$ and $f'(x_0) = 0$.

The last theorem has a generalization based on another $(PS)$ condition introduced by G. Cerami [Ce].

Definition 1.5 (Cerami, 1978). Let $c \in \mathbb{R}$. We say that the $C^1$-functional $f : X \to \mathbb{R}$ satisfies $(PSC)_c$ condition if every sequence $(x_j)_j$ in $X$ such that

$$\lim_j f(x_j) = \text{cand} \lim_j (1 + ||x_j||)||f'(x_j)|| = 0,$$
has a convergent subsequence.

The following minimizing theorem is proved in Ekeland [Ek2], p.139.

**Theorem 1.10.** Let $f : X \to \mathbb{R}$ be a $C^1$-functional bounded from below and $c = \inf f$. If $(PSC)_c$ condition is satisfied, then $c$ is minimum of $f$.

The proof is based on the Ekeland principle applied to the space $X$ equipped by so called geodesic distance. Let $\gamma \in C^1([0,1], X)$ be a curve in $X$. The geodesic length $l(\gamma)$ of the curve $\gamma$ is

$$l(\gamma) = \int_0^1 \frac{\|\dot{\gamma}(t)\|}{1 + \|\gamma(t)\|} \, dt. \quad (1.12)$$

If $x_1$ and $x_2$ are two points of $X$, the geodesic distance $\delta(x_1, x_2)$ between $x_1$ and $x_2$ is defined as

$$\delta(x_1, x_2) = \inf \left\{ l(\gamma) : \gamma \in C^1([0,1], X), \gamma(0) = x_1, \gamma(1) = x_2 \right\}. \quad (1.13)$$

We have

$$\begin{align*}
\delta(x_1, x_2) &\leq \|x_1 - x_2\|, \\
\delta(0, x) & = \int_0^1 \frac{\|x\|}{1 + t \|x\|} \, dt = \ln (1 + \|x\|).
\end{align*}$$

If $x_1$ and $x_2$ belong to a bounded set $B \subset X$, there exists $k > 0$ such that

$$\delta(x_1, x_2) \geq k \|x_1 - x_2\|.$$

**Proof of Theorem 1.10.** By Corollary 1.2 applied to $(X, \delta)$ with $\varepsilon = \frac{1}{j^2}$, we obtain a sequence $(x_j)_j$ such that

$$\inf f \leq f(x_j) \leq \inf f + \frac{1}{j^2}, \quad (1.14)$$

$$f(x) \geq f(x_j) - \frac{1}{j} \delta(x, x_j), \quad \forall x \in X.$$ 

Taking $x = x_j + th$, $t > 0$, $h \in X$, we have

$$f(x_j + th) - f(x_j) \geq -\frac{1}{j} \delta(x_j, x_j + th).$$

By the properties of the geodesic distance and making a change of variable
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\[
\frac{1}{t} (f(x_j + th) - f(x_j)) \geq -\frac{\|h\|}{jt} \int_0^t ds (1 + \|x_j + sh\|).
\]

Letting \( t \to 0 \) in last inequality we get

\[
\langle f'(x_j), h \rangle \geq -\frac{1}{j} (1 + \|x_j\|)^{-1} \|h\|,
\]
or

\[
(1 + \|x_j\|) \|f'(x_j)\| \leq \frac{1}{j}. \tag{1.15}
\]

From (1.14), (1.15) and \((PSC)\) condition, it follows that there is a convergent subsequence of \((x_j)\), which we denote in the same way by \((x_j)\), and \(x_0 \in X\) such that \(\lim_j x_j = x_0\), \(f(x_0) = c\) and \(f'(x_0) = 0\). End Proof

1.2.4 Borwein–Preiss Principle and Second-Order Information for Minimizing Sequences

Let \(X\) be a Banach space and \(\mathcal{F}\) be the class of functionals \(\phi : X \to \mathbb{R}\) of the form

\[
\phi(x) = \frac{1}{2} \sum_{n=1}^{\infty} \lambda_n \|x - x_n\|^2,
\]
where \((x_n)_n\) be a convergent sequence in \(X\) and

\[
\lambda_n \geq 0, \sum_{n=1}^{\infty} \lambda_n = 1.
\]

The generalized Borwein–Preiss variational principle is as follows [BP]

**Theorem 1.11** (Borwein & Preiss, 1987). Let \(f : X \to \mathbb{R}\) be a lower semicontinuous functional bounded below and \(\varepsilon > 0\). If \(x_0 \in X\) is such that

\[
f(x_0) < \inf_{x \in X} f(x) + \varepsilon, \tag{1.16}
\]

then there exist \(x_\varepsilon \in X\) and \(\phi \in \mathcal{F}\) such that

\[
f(x_\varepsilon) < \inf_{x \in X} f(x) + \varepsilon, \tag{1.17}
\]

\[
\|x_\varepsilon - x_0\| \leq 1, \tag{1.18}
\]

\[
f(x) > f(x_\varepsilon) + 2\varepsilon (\phi(x_\varepsilon) - \phi(x)), \quad x \neq x_\varepsilon. \tag{1.19}
\]
Proof. By (1.16) choose positive numbers \( \varepsilon_1, \varepsilon_2, \mu, \theta \) and \( \delta \) such that

\[
f(x_0) - \inf f < \varepsilon_2 < \varepsilon_1 < \varepsilon, \tag{1.20}
\]

\[
0 < \mu < 1 - \frac{\varepsilon_1}{\varepsilon}, \tag{1.21}
\]

\[
0 < \frac{\theta}{\mu} < \frac{\varepsilon_1}{(\sqrt{\varepsilon_1} + \sqrt{\mu \varepsilon_2})^2} < 1, \tag{1.22}
\]

\[
\delta = (1 - \mu) \varepsilon.
\]

We iteratively construct \( \phi \in \mathcal{F} \) as follows. Let \( f_0 := f \),

\[
f_1(x) := f_0(x) + \delta \|x - x_0\|^2,
\]

and recursively

\[
f_{n+1}(x) := f_n(x) + \delta \mu^n \|x - x_n\|^2, \tag{1.23}
\]

where \( x_{n+1} \) is chosen so that

\[
f_{n+1}(x_{n+1}) \leq \theta f_n(x_n) + (1 - \theta) \inf f_n. \tag{1.24}
\]

Denote

\[
s_{n+1} := \inf f_n, \quad a_n := f_n(x_n).
\]

From (1.24) we have

\[
s_n \leq s_{n+1} \leq a_{n+1} \leq \theta a_n + (1 - \theta) s_{n+1} \leq a_n
\]

and then

\[
a_{n+1} - s_{n+1} \leq \theta (a_n - s_n) \leq \theta^{n+1} (a_0 - s_0). \tag{1.25}
\]

Replacing \( x = x_{n+1} \) in (1.23)

\[
a_{n+1} = f_n(x_{n+1}) + \delta \mu^n \|x_{n+1} - x_n\|^2 \\
\geq s_{n+1} + \delta \mu^n \|x_{n+1} - x_n\|^2
\]

which from (1.20) and (1.25) implies
\[\delta \mu^n \|x_{n+1} - x_n\|^2 \leq \theta^{n+1} (a_0 - s_0) \leq \theta^{n+1} \varepsilon_2.\]

From (1.22) it follows that \((x_n)_n\) is a Cauchy sequence because for \(m > n\) we have

\[
\|x_m - x_n\| \leq \|x_{n+1} - x_n\| + \ldots + \|x_m - x_{m-1}\| \\
\leq \left(\frac{\theta}{\mu}\right)^{\frac{1}{2}} \left(\frac{\theta \varepsilon_2}{\delta}\right)^{\frac{1}{2}} + \ldots + \left(\frac{\theta}{\mu}\right)^{\frac{m-1}{2}} \left(\frac{\theta \varepsilon_2}{\delta}\right)^{\frac{1}{2}} \\
= \left(\frac{\theta \varepsilon_2}{\delta}\right)^{\frac{1}{2}} \left(\frac{\theta}{\mu}\right)^{\frac{n}{2}} \frac{1 - \left(\frac{\theta}{\mu}\right)^{\frac{m-n}{2}}}{1 - \left(\frac{\theta}{\mu}\right)^{\frac{1}{2}}} \\
< \left(\frac{\theta \varepsilon_2}{\delta}\right)^{\frac{1}{2}} \frac{1}{1 - \left(\frac{\theta}{\mu}\right)^{\frac{1}{2}}} < \left(\frac{\varepsilon_1}{\delta}\right)^{\frac{1}{2}}.
\]

Then

\[
\|x_m - x_n\| < \left(\frac{\varepsilon_1}{\delta}\right)^{\frac{1}{2}} < 1, \quad (1.26)
\]

since by (1.21)

\[
\frac{\varepsilon_1}{\delta} = \frac{\varepsilon_1}{\varepsilon (1 - \mu)} < \frac{\varepsilon_1}{\varepsilon} = 1.
\]

Let \(x_\varepsilon\) be the limit of the sequence \((x_n)_n\) and

\[
\phi (x) = \frac{1}{2} \sum \mu^n (1 - \mu) \|x - x_n\|^2.
\]

Letting \(m \to \infty\) in (1.26) we obtain

\[
\|x_n - x_\varepsilon\| \leq 1.
\]

By (1.25) and the lower semicontinuity of \(f_n\) we have

\[
f(x) + 2 \varepsilon \phi (x) = \sup f_n (x) \geq \lim_{n} s_n \\
= \lim_{n} f_n (x_n) \geq \sup \liminf f_m (x_n) \\
\geq \sup m f_m (x_\varepsilon) = f(x_\varepsilon) + 2 \varepsilon \phi (x_\varepsilon)
\]
which completes the proof.

End Proof

In the case of Hilbert spaces the class $\mathcal{F}$ in Theorem 1.11 reduces to

$$\mathcal{F}_0 = \left\{ \phi \in \mathcal{F} : \phi(x) = \frac{1}{2} \|x - x_0\|^2 \right\}.$$  

The inequality (1.19) takes the form

$$f(x) > f(x_\epsilon) + \epsilon \left( \|x_\epsilon - x_0\|^2 - \|x - x_0\|^2 \right), \quad \forall x \in X. \quad (1.27)$$

If $f$ is a $C^2$ functional then

$$\frac{1}{t^2} \left( f(x + th) + f(x - th) - 2f(x) \right) \to \langle f''(x) h, h \rangle, \quad (1.28)$$

as $t \to 0$. Writing (1.27) with $x$ replaced by $x_\epsilon + th$ and $x_\epsilon - th$ and adding both inequalities, we obtain

$$f(x_\epsilon + th) + f(x_\epsilon - th) - 2f(x_\epsilon) \geq -2\epsilon t^2 \|h\|^2.$$  

Dividing by $t^2$, letting $t \to 0$ and using (1.28) we have

$$< f''(x_\epsilon) h, h > \geq -2\epsilon \|h\|^2, \quad \forall h \in X,$$

that is,

$$f''(x_\epsilon) \geq -2\epsilon I.$$  

We obtain as a consequence of Theorem 1.11 a second order information about minimizing sequences for functionals in Hilbert spaces (see Fang & Ghoussoub [FG]).

**Corollary 1.4** (Fang & Ghoussoub, 1992). *Let $X$ be a Hilbert space, $f : X \to \mathbb{R}$, be of class $C^2$ and bounded below. Then there exists a sequence $(x_j)_j$ such that*

$$\lim_j f(x_j) = \inf f, \quad \lim_j \|f'(x_j)\| = 0,$$

$$\lim_j \langle f''(x_j) h, h \rangle \geq 0, \quad \forall h \in X.$$
1.3 Deformation Theorems and Palais–Smale Conditions

The original approach to prove the mountain-pass theorem of Ambrosetti & Rabinowitz [ARa] is based on the Deformation lemma, variants of which we consider in this section.

Let $f \in C^1(X, \mathbb{R})$ be a functional defined on the open subset $X$ in the Banach space $E$. We introduce the following notations

\[ K = \{ x \in E : f'(x) = 0 \}, \quad K_c = \{ x \in K(f) : f(x) = c \}, \]
\[ f^- = \{ x \in E : f(x) \leq c \}, \quad f_c = \{ x \in E : f(x) \geq c \}, \]
\[ f_{a}^{b} = \{ x \in E : a \leq f(x) \leq b \} = f_a \cap f_b, \]
\[ B_\rho = \{ x \in E : ||x|| \leq \rho \}, \quad S_\rho = \{ x \in E : ||x|| = \rho \}. \]

and

\[ d(x, F) = \inf \{ ||x - y||, y \in F \}, \quad F_\delta = \{ x \in E : d(x, F) < \delta \}, \]

where $F$ is a closed set in $E$.

A continuous mapping $\eta(t, x) : [0, 1] \times X \to X$ is said to be a homotopy of homeomorphisms if for all $t \in [0, 1]$

1. $\eta(t, x) : X \to X$ is a homeomorphism,
2. $\eta(0, x) = x$, for all $x \in X$.

The following theorem is proved by P.Rabinowitz [Ra2], Theorem A.4.

**Theorem 1.12** (Rabinowitz, 1986). Let $f \in C^1(E, \mathbb{R})$ and satisfy (PS) condition. For any $c \in \mathbb{R}$, $\varepsilon_0 > 0$ and any neighborhood $N$ of $K_c$ there exist $\varepsilon \in (0, \varepsilon_0)$ and $\eta \in C([0, 1] \times E, E)$ such that

1. $\eta(0, x) = x$, for all $x \in E$.
2. $\eta(t, x) = x$ for all $t \in [0, 1]$ if $|f(x) - c| \geq \varepsilon_0$.
3. $\eta(1, f^{c+\varepsilon} \setminus N) \subset f^{c-\varepsilon}$.
4. if $K_c = \emptyset$ then $\eta(1, f^{c+\varepsilon}) \subset f^{c-\varepsilon}$.
5. if $f$ is even, $\eta(t, x)$ is odd in $x$ for all $t \in [0, 1]$.

The proof of Theorem 1.12 is based on the pseudo-gradient vector field due to Palais [Pal].

**Definition 1.6** (Palais, 1970). Let $X \subset E$ be an open subset, $f \in C^1(X, \mathbb{R})$ and $0 < \alpha < \beta$ be given. A pseudo-gradient vector field for $f$ on $X \setminus K$ is a locally Lipschitz continuous mapping $V : X \setminus K \to E$ such that

1. $\|V(x)\| \leq \beta \|f'(x)\|$, 
2. $\alpha \|f'(x)\|^2 \leq \langle f'(x), V(x) \rangle$. 

We have

**Lemma 1.1.** Under the assumptions of Definition 1.6, there exists a pseudo-gradient vector field for $f$ on $X \setminus K$.

In the case $\alpha = 1$, $\beta = 2$, Lemma 1.1 is proved in Rabinowitz [Ra2], Willem [Wil1]. In [Ra2], Appendix A is given the idea for general $\alpha$ and $\beta$. Further comments on pseudo-gradient vector fields are given in Ramos [Ram].

**Proof of Lemma 1.1.** Since $\frac{2\alpha}{\alpha + \beta} < 1$, if $x \in X \setminus K$ and by the definition of norm in $E^*$, there exists $w_x \in E$, $\|w_x\| = 1$ such that

$$\langle f'(x), w_x \rangle > \frac{2\alpha}{\alpha + \beta} \|f'(x)\|.$$ 

Let $V_x := \frac{\alpha + \beta}{2} \|f'(x)\| w_x$. Then

$$\|V_x\| < \beta \|f'(x)\|,$$

$$\langle f'(x), V_x \rangle > \alpha \|f'(x)\|^2.$$ 

Since $f'$ is continuous there exists an open neighborhood $U_x$ of $x$ such that for every $y \in U_x$

$$\|V_x\| \leq \beta \|f'(y)\|,$$

$$\langle f'(y), V_x \rangle \geq \alpha \|f'(x)\|^2.$$ (1.29) (1.30)

The family $\{U_x : x \in X \setminus K\}$ is an open covering of $X \setminus K$ and let $\{W_i\}$ be a locally finite refinement of $\{U_x\}$ and $\rho_i(x) = d(x, X \setminus W_i)$. For each $i$, let $x_i$ be such that $W_i \subset U_{x_i}$ and put $V_i = V_{x_i}$. The function $\rho_i(x)$ is Lipschitz continuous and $\rho_i(x) = 0$ if $x \notin W_i$. The sum $\sum \rho_i(x)$ is locally finite and $\sum \rho_i(x) > 0$ for all $x \in X \setminus K$. Define for $x \in X \setminus K$

$$V(x) := \frac{1}{\sum \rho_i(x)} \sum \rho_i(x) V_i.$$ 

The mapping $V : X \setminus K \rightarrow E$ is a locally Lipschitz continuous and since $V(x)$ is a convex combination of vectors satisfying (1.29) and (1.30) we have
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\[ \|V(x)\| \leq \frac{1}{\sum_i \rho_i(x)} \sum_i \rho_i(x) \|V_i\| \leq \beta \|f'(x)\| \]

and

\[ \left\langle f'(x), V(x) \right\rangle = \frac{1}{\sum_i \rho_i(x)} \sum_i \rho_i(x) \left\langle f'(x), V_i(x) \right\rangle \geq \alpha \left\|f'(x)\right\|^2. \]

End Proof

Corollary 1.5. Given \( 0 < \alpha < \beta \) there exist locally Lipschitz mappings \( V_j : X \setminus K \to E, j = 1, 2 \) such that for all \( x \in X \setminus K \)
1. \( \|V_1(x)\| \leq \beta \) and \( \alpha \left\|f'(x)\right\| \leq \left\langle f'(x), V_1(x) \right\rangle \),
2. \( \alpha \leq \left\langle f'(x), V_2(x) \right\rangle \leq \left\|f'(x)\right\| \|V_2(x)\| \leq \beta. \)

Proof. Let \( x \in X \setminus K \) and \( w_x \in E, \|w_x\| = 1 \) be such that

\[ \left\langle f'(x), w_x \right\rangle > \frac{2\alpha}{\alpha + \beta} \left\|f'(x)\right\|. \]

Let \( V_{1,x} := \frac{\alpha + \beta}{2} w_x \) and \( V_{2,x} := \frac{\alpha + \beta}{2\|f'(x)\|} w_x. \) Then we have

\[ \alpha \left\|f'(x)\right\| < \left\langle f'(x), V_{1,x} \right\rangle, \quad \|V_{1,x}\| < \beta, \]
\[ \alpha < \left\langle f'(x), V_{2,x} \right\rangle \leq \left\|f'(x)\right\| \|V_{2,x}\| < \beta. \]

Since \( f' \) is continuous, there exists an open neighborhood \( U_x \) of \( x \) such that for \( y \in U_x \)

\[ \alpha \left\|f'(x)\right\| < \left\langle f'(y), V_{1,x} \right\rangle, \]
\[ \alpha < \left\langle f'(y), V_{2,x} \right\rangle \leq \left\|f'(y)\right\| \|V_{2,x}\| < \beta. \]

The proof can be completed by following the proof of Lemma 1.1. End Proof

We say that a homotopy of homeomorphisms \( \eta(t,x) : [0,1] \times X \to X \) is \( f \)-decreasing (\( f \)-increasing) if whenever \( 0 \leq t_1 \leq t_2 \leq 1 \) then

\[ f(\eta(t_1,x)) \geq f(\eta(t_2,x)), \quad (f(\eta(t_1,x)) \leq f(\eta(t_2,x))) \quad \forall x \in X. \]

Theorem 1.12 was generalized by M. Willem [Wil1], [Wil2], Bartolo, Benci & Fortunato [BBF], Chang [Ch3], Brezis & Nirenberg [BN] in various directions.
Theorem 1.13. Let \( f \in C^1(X, \mathbb{R}) \) and \( F \) and \( G \) be closed disjoint subsets of \( X \). Let \( c \in \mathbb{R}, \varepsilon \) and \( \delta > 0 \) be numbers such that \( F_{2\delta} \cap G = \emptyset \) and

\[
x \in f^{-1}[c-\varepsilon, c+\varepsilon] \cap F_{2\delta} \Rightarrow \|f'(x)\| \geq \frac{4\varepsilon}{\delta}.
\]

Then there exists a \( f \)-decreasing homotopy of homeomorphisms \( \eta : [0,1] \times X \to X \) such that:

1. \( \eta(t,x) = x \) if either \( x \in G \) or \( |f(x) - c| \geq 2\varepsilon \),
2. \( \eta(1, f^{-1}[c-\varepsilon, c+\varepsilon] \cap F_{2\delta}) \subset f^{-1}[c-\varepsilon, c+\varepsilon] \cap F_{2\delta} \),
3. \( \|\eta(t,x) - x\| \leq 2\delta t \).

Proof. Consider the sets

\[
A := \{x : |f(x) - c| \geq 2\varepsilon\} \cup \left\{x : \|f'(x)\| \leq \frac{2\varepsilon}{\delta}\right\} \cup G,
\]

\[
B := f^{-1}[c-\varepsilon, c+\varepsilon] \cap F_{2\delta},
\]

and define the function

\[
\chi(x) := \frac{d(x, A)}{d(x, A) + d(x, B)}.
\]

Let \( V : X \setminus K \to E \) be a locally Lipschitz mapping, satisfying Corollary 1.5 (2), with \( \alpha = 1, \beta = 2 \) and let

\[
g(x) := \chi(x) V(x).
\]

Consider the Cauchy problem

\[
\begin{cases}
\dot{\sigma}(t) = -g(\sigma(t)), \\
\sigma(0) = x,
\end{cases}
\]

for every \( x \in X \). We have \( g(x) = 0 \) if \( x \in A \). If \( x \notin A \) and \( x \in X \setminus K \), then

\[
\|g(x)\| \leq \|V(x)\| \leq \frac{2}{\|f'(x)\|} \leq \frac{\delta}{\varepsilon}. 
\]

By the fundamental existence-uniqueness theorem for ordinary differential equations in Banach spaces (see Cartan [Car], Ramos [Ram]), the problem (1.31) has a unique solution \( \sigma(.,x) : \mathbb{R}^+ \times X \to E \) and \( \sigma(t,.) : X \to X \) is a homeomorphism.
The homotopy $\sigma(t, x)$ is $f$-decreasing because
\[ \frac{d}{dt} f(\sigma(t, x)) = \langle f'(\sigma(t, x)), \dot{\sigma}(t, x) \rangle \]
\[ = -\chi(\sigma(t, x)) \langle f'(\sigma(t, x)), V(\sigma(t, x)) \rangle \]
\[ \leq -\chi(\sigma(t, x)) \leq 0. \]

Let $\eta(t, x) = \sigma(2\varepsilon t, x)$. Since $\chi(x) = 0$ if $x \in A$, then $\eta(t, x) = x$ if $x \in G$ or $|f(x) - c| \geq 2\varepsilon$, so (1) is proved. We get (3) by (1.32)
\[ \| \eta(t, x) - x \| = \| \sigma(2\varepsilon t, x) - \sigma(0, x) \| \]
\[ \leq \int_0^{2\varepsilon t} \| \dot{\sigma}(s) \| \, ds = \int_0^{2\varepsilon t} \| g(\sigma(s)) \| \, ds \]
\[ \leq \frac{\delta}{\varepsilon} 2\varepsilon t = 2\delta t. \]

Let us prove (2). From (3) it follows that $\eta(t, F) \subset F_{2\delta}$ for all $t \in [0, 1]$. Let $x \in f_{c+\varepsilon} \cap F$. If there exists $t_0 \in [0, 1]$ such that $f(\sigma(2\varepsilon t_0, x)) \leq c - \varepsilon$, then
\[ f(\sigma(2\varepsilon, x)) \leq f(\sigma(2\varepsilon t_0, x)) \leq c - \varepsilon \]
and the assertion follows.
If $f(\sigma(2\varepsilon t, x)) > c - \varepsilon$ for every $t \in [0, 1]$, since
\[ c + \varepsilon \geq f(x) = f(\sigma(0, x)) \geq f(\sigma(2\varepsilon, x)) > c - \varepsilon, \]
we have $\sigma(2\varepsilon, x) \in f^{-1}[c - \varepsilon, c + \varepsilon] \cap F_{2\delta} = B$. Then by Corollary 1.5 (2)
\[ f(\sigma(2\varepsilon, x)) - f(x) = \int_0^{2\varepsilon} \frac{d}{ds} f(\sigma(s, x)) \, ds \]
\[ = \int_0^{2\varepsilon} \langle f'(\sigma(s, x)), \dot{\sigma}(s, x) \rangle \, ds \]
\[ = -\int_0^{2\varepsilon} \langle f'(\sigma(s, x)), V(\sigma(s, x)) \rangle \, ds \]
\[ \leq -2\varepsilon \]
and so
\[ c - \varepsilon < f(\sigma(2\varepsilon, x)) \leq f(x) - 2\varepsilon \leq c + \varepsilon - 2\varepsilon = c - \varepsilon, \]
which is a contradiction. End Proof
Corollary 1.6. Let \( f \) satisfy \((PS)_c\) condition and \( N = K_{c,\delta} \) be a neighborhood of \( K_c \). Then there exist \( \varepsilon > 0 \) and a \( f \)-decreasing homotopy of homeomorphisms \( \eta(t,x) : [0,1] \times X \to X \) such that

1. \( \eta(t,x) = x \) if \( x \in K_c \) or \( |f(x) - c| \geq 2\varepsilon \),
2. \( \eta(1, f^{c+\varepsilon} \setminus N) \subset f^{c-\varepsilon} \),
3. \( \|\eta(t,x) - x\| \leq 2\delta t \).

Proof. There exist \( \varepsilon_0 > 0 \) and \( \beta > 0 \) such that

\[
|f(x) - c| \leq \varepsilon_0, \quad d(x, K_c) \geq 2\delta \implies \|f'(x)\| \geq \beta.
\]

Otherwise, there exists a sequence \((x_j)\) such that

\[
|f(x_j) - c| \leq \frac{1}{j}, \quad d(x_j, K_c) \geq 2\delta, \quad \|f'(x_j)\| < \frac{1}{j}.
\]

By \((PS)_c\) condition there exists a subsequence of \((x_j)\), which we denote again by \((x_j)\) such that \( \lim_j x_j = x_0 \), \( f(x_0) = c \) and \( f'(x_0) = 0 \) which contradicts to \( d(x_0, K_c) \geq 2\delta \).

Let \( 0 < \varepsilon < \min(\varepsilon_0, \beta\delta/4) \). Then

\[
|f(x) - c| \leq \varepsilon, \quad d(x, K_c) \geq 2\delta \implies \|f'(x)\| \geq \frac{4\varepsilon}{\delta}.
\]

The assertion follows from Theorem 1.13 taking \( G = K_c \) and \( F = X \setminus K_{c,\delta} \).

End Proof.

Corollary 1.7. Let \( f \) satisfies \((PS)_c\) condition and \( K_c = \emptyset \). Then there exist \( \varepsilon > 0 \) and a \( f \)-decreasing homotopy of homeomorphisms \( \eta : [0,1] \times X \to X \) such that:

1. \( \eta(t,x) = x \) if \( |f(x) - c| \geq 2\varepsilon \),
2. \( \eta(1, f^{c+\varepsilon} \setminus N) \subset f^{c-\varepsilon} \).

Proof. By \((PS)_c\) condition there exist \( \varepsilon_0, \beta > 0 \) such that

\[
|f(x) - c| \leq \varepsilon_0 \implies \|f'(x)\| \geq \beta.
\]

Otherwise, there exists a sequence \((x_j)\) such that

\[
|f(x_j) - c| \leq \frac{1}{j} \quad \text{and} \quad \|f'(x_j)\| \leq \frac{1}{j}.
\]

By \((PS)_c\) condition it follows that \( c \) is a critical value, which contradicts to \( K_c = \emptyset \). Let \( \delta > 0 \) and \( \varepsilon \in (0, \min(\varepsilon_0, \beta\delta/4)) \). The assertion follows from Theorem 1.13 taking \( G = \emptyset \) and \( F = X \).

End Proof.
We introduce another (PS) type condition which generalizes (PSC)_c condition.

Let \( \Phi \) be the set of positive increasing functions \( \varphi(s) : \mathbb{R}^+ \to \mathbb{R}^+ \), \( \varphi(0) > 0 \) such that \( \varphi(s) = O(|s|) \) as \( |s| \to \infty \), i.e. there are \( C > 0, R > 0 \) such that
\[
\varphi(s) \leq Cs \quad \text{if} \quad s \geq R.
\]

**Definition 1.7.** Let \( \varphi \in \Phi \) and \( c \in \mathbb{R} \). The \( C^1 \) functional \( f \) satisfies (PS)_{c, \varphi} condition for a given \( \varphi \in \Phi \) if every sequence \( (x_j) \) such that
\[
\lim_j f(x_j) = c \quad \text{and} \quad \lim_j \varphi(\|x_j\|) \left\| f'(x_j) \right\| = 0,
\]
has a convergent subsequence.

If \( \varphi(s) = 1 + s, s \geq 0 \), we are in the case of (PSC)_c condition.

Next theorem extends a result of Ramos [Ram], Theorem 2.12.

**Theorem 1.14.** Suppose that \( f \in C^1(X, \mathbb{R}) \) satisfies (PS)_{c, \varphi} condition for a given \( c \in \mathbb{R} \) and \( \varphi \in \Phi \). Let \( N = K_{c, \delta} \) be a neighborhood of \( K_c \).

Then there exist \( \varepsilon > 0 \) and a \( f \)-decreasing homotopy of homeomorphisms \( \eta : [0,1] \times X \to X \) such that:
1. \( \eta(t, x) = x \) if \( x \in K_c \) or \( |f(x) - c| \geq 2\varepsilon \),
2. \( \eta(1, f^{-c+\varepsilon} \setminus N) \subset f^{-c-\varepsilon} \).

**Proof.** The set \( N \) is bounded because \( K_c(f) \) is compact by (PS)_{c, \varphi} condition. Let \( R_0 \) be such that \( N \subset B_{R_0} \). There exist numbers \( \varepsilon_0 > 0, R > R_0, c_0 > 0 \) and \( \beta > 0 \) such that if
\[
|f(x) - c| \leq 2\varepsilon_0, \quad \|x\| \geq R,
\]
then
\[
\left\| f'(x) \right\| \geq \frac{\beta}{\varphi(\|x\|)} \geq \frac{\beta c_0}{\|x\|}.
\]

(1.33)

Otherwise, for every \( j > 0 \) there exists \( x_j \) such that
\[
|f(x_j) - c| \leq \frac{1}{j}, \quad \|x_j\| \geq j, \quad \varphi(\|x_j\|) \left\| f'(x_j) \right\| < \frac{1}{j}.
\]

By (PS)_{c, \varphi} condition \( (x_j)_j \) has a convergent subsequence, which is a contradiction to \( \lim_j \|x_j\| = \infty \). Moreover, there exists \( \varepsilon_1 < \varepsilon_0 \) such that if
\[
|f(x) - c| \leq 2\varepsilon_1, \quad \|x\| \leq R, \quad d(x, K_c) \geq \delta,
\]
then
\[ \|f'(x)\| \geq \frac{4\varepsilon_1}{\delta}. \] (1.34)

In fact, assuming the contrary, there exists a sequence \((x_j)_j\) such that
\[ |f(x_j) - c| \leq \frac{1}{j}, \quad \|x_j\| \leq R, \]
\[ d(x_j, K_c) \geq \delta \quad \text{and} \quad \|f'(x_j)\| < \frac{1}{j}. \]

Since \(\varphi\) is increasing, \(\varphi (\|x_j\|) \leq \varphi (R).\) So
\[ \varphi (\|x_j\|) \|f'(x_j)\| \leq \varphi (R) \|f'(x_j)\| < \frac{\varphi (R)}{j}, \]
and it follows that
\[ \lim_j \varphi (\|x_j\|) \|f'(x_j)\| = 0. \]

Therefore, by \((PS)_{c, \varphi}\) condition, \((x_j)_j\) has a convergent subsequence which we still denote by \((x_j)_j\). Then \(\lim_j x_j = x_0, \quad f'(x_0) = 0\) and \(f(x_0) = c\), which contradicts to \(d(x, K_c) \geq \delta\).

Let \(0 < \varepsilon < \varepsilon_1\). As in the proof of Theorem 1.13 we define
\[
A = \{ x : |f(x) - c| \geq 2\varepsilon \} \cup \{ x : d(x, K_c) \leq \delta \} \cup K, \\
B = f^{-1}[c - \varepsilon, c + \varepsilon] \cap \{ x : d(x, K_c) \geq 2\delta \},
\]
and
\[ \chi(x) = \frac{d(x, A)}{d(x, A) + d(x, B)}. \]

Let \(\alpha = 1, \beta = 2\) and consider a locally Lipschitz mapping \(V : X \setminus K \to E\), according to Corollary 1.5, (2). Define \(g(x) = \chi(x) V(x)\). We have \(g(x) = 0\) if \(x \in A\) and if \(x \notin A\), by (1.33) and (1.34),
\[ \|g(x)\| \leq \|V(x)\| \leq \frac{2}{\|f'(x)\|} \leq \frac{\delta}{2\varepsilon} + \frac{2}{\beta c_0} \|x\|. \quad (1.35) \]

By (1.35) for every \(x \in X\), the Cauchy problem
\[
\begin{align*}
\dot{\sigma}(t) &= -g(\sigma(t)), \\
\sigma(0) &= x.
\end{align*}
\]
has a unique solution $\sigma(., x) : \mathbb{R}^+ \to E$.

Let $\eta(t, x) = \sigma(2\varepsilon t, x), 0 \leq t \leq 1$. Since $\chi(x) = 0$ if $x \in A$ the assertion (1) of theorem is satisfied.

Let us prove (2), which means that for every $x$ such that

$$f(x) \leq c + \varepsilon, \quad d(x, K_c) \geq 4\delta,$$

we have

$$f(\eta(1, x)) \leq c - \varepsilon.$$

By contradiction, assume there exists $y$ such that

$$f(y) \leq c + \varepsilon, \quad d(y, K_c) \geq 4\delta, \quad f(\eta(1, y)) > c - \varepsilon.$$

Then $f(y) = f(\eta(0, y)) \geq f(\eta(1, y)) > c - \varepsilon$ and $y \in B$. However $\sigma(2\varepsilon t, y)$ cannot stay in $B$ for every $t \in [0, 1]$. Otherwise $d(\sigma(2\varepsilon t, y), K_c) \geq 2\delta$ for all $t \in [0, 1]$ and

$$c - \varepsilon < f(\sigma(2\varepsilon, y)) \leq f(y) - \int_0^{2\varepsilon} \left\langle f'(\sigma(s)), V(\sigma(s)) \right\rangle ds \leq c + \varepsilon - 2\varepsilon = c - \varepsilon,$$

which is a contradiction.

Since

$$c + \varepsilon \geq f(y) \geq f(\sigma(2\varepsilon t, y)) \geq f(\sigma(2\varepsilon, y)) > c - \varepsilon,$$

there exist $0 \leq t_1 \leq t_2 \leq 1$ such that

$$d(\sigma(2\varepsilon t_1, y), K_c) = 4\delta \geq d(\sigma(2\varepsilon t, y), K_c) \geq 2\delta = d(\sigma(2\varepsilon t_2, y), K_c)$$

for every $t \in [t_1, t_2]$. We have $\sigma(2\varepsilon [t_1, t_2], y) \subset B \cap B_R$. Therefore, by Corollary 1.5, (2) and (1.34), we have

$$2\delta \leq \|\sigma(2\varepsilon t_2) - \sigma(2\varepsilon t_1)\| \leq \int_{2\varepsilon t_1}^{2\varepsilon t_2} ||\dot{\sigma}(s)|| ds \leq \int_{2\varepsilon t_1}^{2\varepsilon t_2} ||V(\sigma(s))|| ds$$
\[
\int_{2t_1}^{2t_2} \frac{2ds}{\|f'(\sigma(s))\|} \leq 2\varepsilon (t_2 - t_1) \frac{2\delta}{4\varepsilon_1} \leq 4\varepsilon \frac{\delta}{4\varepsilon_1} \leq \delta,
\]

which is a contradiction. Therefore (2) is satisfied and the Theorem is proved. End Proof.

Another \((PS)\) type condition has been introduced by M. Schechter [Sch1], [Sch2].

Let \(\Psi\) be the set of positive non-increasing functions \(\psi(s) : \mathbb{R}^+ \to \mathbb{R}^+\) such that

\[
\int_{1}^{\infty} \psi(s) \, ds = \infty.
\]

**Definition 1.8** (Schechter, 1991). Let \(c \in \mathbb{R}\) and \(\psi \in \Psi\). The \(C^1\) functional \(f\) satisfies \((PS)_{c,\psi}\) if every sequence \((x_j)_j \subset X\) such that

\[
\lim_j f(x_j) = c \quad \text{and} \quad \lim_j \frac{\|f'(x_j)\|}{\psi(\|x_j\|)} = 0,
\]

has a convergent subsequence in \(X\).

If \(\psi\) is a constant we have the usual \((PS)_c\) condition, if \(\psi(s) = \frac{1}{1+s}\) we are in the case of Cerami \((PSC)_c\) condition. If \(\phi \in \Phi\), then \(\frac{1}{\phi} \in \Psi\).

**Theorem 1.15.** Let \(f\) satisfy \((PS)_{c,\psi}\) condition for some \(\psi \in \Psi\) and \(c \in \mathbb{R}\) be such that \(K_c = \emptyset\). Take \(M > 0\). Then there exist \(\varepsilon > 0\) and a \(f\)-increasing homotopy of homeomorphisms \(\eta : [0,1] \times X \to X\) such that

1. \(\eta(t,x) = x\) if \(|f(x) - c| \leq 2\varepsilon\) for \(t \in [0,1]\)
2. \(\eta(1,f_{c-\varepsilon} \cap B_M) \subset f_{c+\varepsilon}\).

**Proof.** There exist \(\varepsilon_0 > 0, \beta > 0\) such that

\[
|f(x) - c| \leq \varepsilon_0 \Rightarrow \|f'(x)\| \geq \beta \psi(\|x\|).
\]

Otherwise, there exists a sequence \((x_j)_j\) such that

\[
|f(x_j) - c| \leq \frac{1}{j}, \quad \|f'(x_j)\| \leq \frac{1}{j} \psi(\|x_j\|).
\]

By \((PS)_{c,\psi}\) condition \(\lim_j x_j = x_0\) and so \(f(x_0) = c, f'(x_0) = 0\), which is a contradiction to \(K_c = \emptyset\).
Let $\varepsilon < \min\{\varepsilon_0, \beta\}$. Then

$$|f(x) - c| \leq \varepsilon \Rightarrow \left\|f'(x)\right\| \geq \varepsilon \psi(\|x\|).$$

Let $T > 0$ be such that

$$4 < \int_M^{M + 2T} \psi(s)\, ds$$

and

\[
A = \{x : |f(x) - c| \geq 2\varepsilon\} \cup K, \\
B = \{x : |f(x) - c| \leq \varepsilon\}, \\
\chi(x) = \frac{d(x, A)}{d(x, A) + d(x, B)}.
\]

By Corollary 1.5, consider $V : X \setminus K \to E$, a locally Lipschitz mapping such that

$$\left\|f'(x)\right\| \leq \left\langle f'(x), V(x)\right\rangle, \quad \|V(x)\| \leq 2.$$ Let $g(x) = \chi(x) V(x)$ and $\sigma(t, x)$ be the solution of Cauchy problem

\[
\begin{cases}
\dot{\sigma}(t) = g(\sigma(t)), \\
\sigma(0) = x.
\end{cases}
\]

Since $\|g(u)\| \leq 2$ the solution $\sigma(t, x)$ of the above problem is defined for every $t \in \mathbb{R}^+$. We have

$$\|\sigma(t, x) - x\| \leq \int_0^t \|\dot{\sigma}(s)\|\, ds \leq \int_0^t \|V(\sigma(s))\|\, ds \leq 2t,$$

and

$$\frac{d}{dt} f(\sigma(t, x)) = \left\langle f'(\sigma(t, x)), \chi(\sigma(t, x)) V(\sigma(t, x))\right\rangle$$

$$= \chi(\sigma(t, x)) \left\langle f'(\sigma(t, x)), V(\sigma(t, x))\right\rangle \geq 0.$$

Therefore, for each $x \in X \setminus K$, the mapping $t \to \sigma(t, x)$ is $f$-increasing.

Moreover $\sigma(t, x) = x$ if $|f(x) - c| \geq 2\varepsilon$, because $\chi|_A = 0$. Then (1) is proved.

Suppose that there exists $y$ with $\|y\| \leq M$ and $f(y) \geq c - \varepsilon$, such that for every $t \in [0, T]$, $f(\sigma(t, y)) < c + \varepsilon$.

Then $\sigma(t, y) \in B$ for $t \in [0, T]$ and
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\[ f(\sigma(T, y)) - f(y) = \int_0^T \frac{d}{ds} f(\sigma(s, y)) \, ds \]
\[ = \int_0^T \left( f'(\sigma(s, y)), \dot{\sigma}(s, y) \right) \, ds \]
\[ = \int_0^T \left( f'(\sigma(s, y)), V(\sigma(s, y)) \right) \, ds \]
\[ \geq \int_0^T \left\| f'(\sigma(s, y)) \right\| \, ds \]
\[ \geq \varepsilon \int_0^T \psi(\|\sigma(s, y)\|) \, ds \]
\[ \geq \varepsilon \int_0^T \psi(\|y\| + 2s) \, ds \]
\[ \geq \varepsilon \int_0^T \psi(M + 2s) \, ds = \frac{\varepsilon}{2} \int_M^{M+2T} \psi(s) \, ds \]
\[ > 2\varepsilon. \]

So, we have

\[ c + \varepsilon > f(\sigma(T, y)) > f(y) + 2\varepsilon \geq c - \varepsilon + 2\varepsilon = c + \varepsilon, \]

which is a contradiction.

Therefore, for every \( x \) satisfying \( \|x\| \leq M \) and \( f(x) \geq c - \varepsilon \), there exists \( t_1 \in [0, T] \) such that \( f(\sigma(t_1, x)) \geq c + \varepsilon \). Then

\[ f(\sigma(T, x)) \geq f(\sigma(t_1, x)) \geq c + \varepsilon \]

and we can take \( \eta(t, x) = \sigma(Tt, x), 0 \leq t \leq 1 \), which completes the proof. End Proof.

In an analogous way, we can prove the following variant result

**Theorem 1.15’.** Let \( f \) satisfies \((PS)_{c, \psi}\) condition for some \( \psi \in \Psi \) and \( c \in \mathbb{R} \) such that \( K_c = \emptyset \). Take \( M > 0 \). Then there exist \( \varepsilon > 0 \) and a \( f \)-decreasing homotopy of homeomorphisms \( \eta : [0, 1] \times X \rightarrow X \) such that

1. \( \eta(t, x) = x \) if \( |f(x) - c| \geq 2\varepsilon \) for \( t \in [0, 1] \)
2. \( \eta(1, f^{c+\varepsilon} \cap B_M(0)) \subset f^{c-\varepsilon} \).

Finally we consider a \((PS)\) condition in scales of Banach spaces, introduced by Struwe [St1], [St2], cf. also Silva [EAS], Li & Willem [LW].

Let \( E \) be a Banach space and

\[ E_1 \subset E_2 \subset \ldots \subset E_n \subset \ldots \subset E \]
be a scale of Banach spaces such that $\bigcup_{n=1}^{\infty} E_n$ is dense in $E$ and
\[
\|u\|_{n+1} \leq \|u\|_n, \quad \forall u \in E_n,
\]
where $\|\cdot\|_n$ denotes the norm in $E_n$.

Let $f : E \to \mathbb{R}$ be a functional, such that $f_n \in C^1(E_n, \mathbb{R})$ for every $n$ where $f_n = f|_{E_n}$.

Note that $\cdots \to E_{n+1}^* \overset{r_n}{\longrightarrow} E_n^* \to \cdots \to E_2^* \overset{i_1}{\longrightarrow} E_1^*$,

where $r_n : E_{n+1}^* \to E_n^*$ is the restriction defined by
\[
\langle r_n(p), x \rangle_n = \langle p, i_n x \rangle_{n+1}, \quad \forall p \in E_{n+1}^*,
\]
where $\langle \cdot, \cdot \rangle_n$ is the pairing between $E_n^*$ and $E_n$ and $i_n : E_n \to E_{n+1}$ is the inclusion mapping. We have
\[
\|r_n p\|_n \leq \|p\|_{n+1}
\]
if $p \in E_{n+1}^*$. Moreover, if $n \geq n_0$ and if $u$ and $v \in E_{n_0}$ we have
\[
\langle f'_{n_0}(u), v \rangle_{n_0} = \lim_{t \to 0} \frac{f_{n_0}(u + tv) - f_{n_0}(u)}{t}
= \lim_{t \to 0} \frac{f_n(u + tv) - f_n(u)}{t}
= \langle f'_n(u), v \rangle_n.
\]
Therefore $f'|_{E_{n_0}} = f'|_{n_0}$. Moreover
\[
\left\| f'_n(u) \right\|_n \geq \left\| f'_n(u) \right\|_{n_0} = \left\| f'|_{E_{n_0}}(u) \right\|_{n_0} = \left\| f'_n(u) \right\|_{n_0}.
\] (1.36)

**Definition 1.9.** Let $c \in \mathbb{R}$ and $f \in C^1(E, \mathbb{R})$. The functional $f$ satisfies $(PS)^*_c$ condition if every sequence $(u_n)_n \subset E$ such that $u_n \in E_n$ and
\[
\lim f_n(u_n) = c \quad \text{and} \quad \lim_n \left\| f'_n(u_n) \right\|_n = 0,
\]
has a strongly convergent subsequence in $E$.

**Definition 1.9’.** Let $f \in C^1(E, \mathbb{R})$. The functional $f$ satisfies \((PS)^*\) condition if every sequence $(u_n)_n \subset E$ such that $u_n \in E_n$ and
\[
\sup_n f_n(u_n) < \infty \quad \text{and} \quad \lim_n \|f_n'(u_n)\|_n = 0,
\]
has a strongly convergent subsequence in $E$.

We denote
\[
K^n_c := \{ x \in E_n : f^n(x) = c, \quad f_n'(x) = 0 \} = E_n \cap K_c.
\]

**Theorem 1.17.** Let $c \in \mathbb{R}$, $\rho > 0$ and $f \in C^1(E, \mathbb{R})$ be a functional. Suppose that $f$ satisfies \((PS)^*_c\) condition and $N$ is a neighborhood of $K_c$. Then there exist $\varepsilon_0 > 0$, $n_0 \in \mathbb{N}$ and a $f$-decreasing homotopy of homeomorphisms
\[
\eta_n : [0, 1] \times E_n \rightarrow E_n
\]
for $n \geq n_0$ such that if $\varepsilon \in (0, \varepsilon_0)$
\begin{enumerate}
  
  \item $\eta_n(t, u) = u$ if either $u \in K^n_c$ or $|f^n(u) - c| \geq 2\varepsilon$,
  
  \item $\eta_n(1, (f^n + \varepsilon \cap N) \cap E_n) \subset f^n - \varepsilon$,
  
  \item $\|\eta_n(t, u) - u\|_n \leq \rho, \quad \forall (t, u) \in [0, 1] \times E_n$.
\end{enumerate}

Proof. The condition \((PS)^*_c\) implies the existence of $\beta > 0$, $\varepsilon_0 > 0$, $\delta \in (0, \rho/2)$ and $n_0$ such that if $n \geq n_0$ and
\[
|f^n(u) - c| \leq \varepsilon_0, \quad u \in (E_n \cap N)_\beta.
\]
then
\[
\|f_n'(u)\|_n \geq \beta. \tag{1.37}
\]
It suffices to choose $\varepsilon \in (0, \min(\varepsilon_0, \beta/4))$ and applying Theorem 1.13 with $X = E_n$, $G = K^n_c$ and $F = E_n \cap N$. End Proof

### 1.4 Mountain-Pass Theorems

In critical point theory, minimax theorems characterize a critical value $c$ of a functional $f : X \rightarrow \mathbb{R}$ as a minimax over a suitable class of sets $A$
\[
c = \inf_{A \in A} \max_{x \in A} f(x).
\]

We state the mountain-pass theorem due to Ambrosetti & Rabinowitz [ARa].
Theorem 1.18 (Ambrosetti & Rabinowitz, 1973). Let $X$ be a real Banach space and $f \in C^1(X, \mathbb{R})$. Suppose that $f$ satisfies $(PS)$ condition, $f(0) = 0$ and

(i) there exist constants $\rho > 0$ and $\alpha > 0$ such that $f(x) \geq \alpha$ if $\|x\| = \rho$,

(ii) there is $e \in X$, $\|e\| > \rho$, such that $f(e) \leq 0$.

Then $f$ has a critical value $c \geq \alpha$ which can be characterized as

$$c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} f(\gamma(t)),$$

where

$$\Gamma = \{ \gamma \in C([0,1] \times X) : \gamma(0) = 0, \gamma(1) = e \}.$$  \hspace{1cm} (1.39)

Geometrically, when $X = \mathbb{R}^2$ the assumptions (i) and (ii) mean that the origin lies in a valley surrounded by a “mountain”

$$\Gamma_f = \{(x, f(x)) \in \mathbb{R}^3 : x \in \mathbb{R}^2 \}.$$  \hspace{1cm} (1.39)

So, there must exist a mountain pass joining $(0,0)$ and $(e, f(e))$ that contains a critical value.

Note that $(PS)$ condition is essential in Theorem 1.18 as the following example shows.

Example 1.3. The function $h(x, y) = x^2 + (x + 1)^3 y^2$ satisfies assumptions (i) and (ii) of Theorem 1.18 but does not satisfy $(PS)$ condition and its unique critical point is $(0,0)$.

Proof. The point $(0,0)$ is a strict local minima and the unique critical point. If $(PS)$ condition is satisfied then $(PS)_c$, with $c > 0$ defined by (1.38), is also satisfied. Let $(x_j, y_j)_j$ be a sequence such that

$$\lim_j \left( x_j^2 + (x_j + 1)^3 y_j^2 \right) = c > 0,$$  \hspace{1cm} (1.40)

$$\lim_j \left( 2x_j + 3(x_j + 1)^2 y_j^2 \right) = 0,$$

$$\lim_j 2(x_j + 1)^3 y_j = 0.$$  

Suppose that $\lim_j (x_j, y_j) = (x_0, y_0) \neq (0,0)$. Passing to the limit in (1.40) we obtain

$$x_0^2 + (x_0 + 1)^3 y_0^2 = c > 0,$$

$$2x_0 + 3(x_0 + 1)^2 y_0^2 = 0,$$

$$2(x_0 + 1)^3 y_0 = 0.$$
which implies a contradiction. End Proof

We give for completeness the proof of Theorem 1.18 based on the deformation approach.

Proof of Theorem 1.18. Suppose by contradiction that $K_c = \emptyset$. Take $\varepsilon$ such that $0 < \varepsilon < \frac{\alpha}{2}$. From (i) and (ii) we have $c \geq \alpha > 2\varepsilon$ and let $\gamma \in \Gamma$ be such that

$$\max_{t \in [0,1]} f(\gamma(t)) < c + \varepsilon. \quad (1.41)$$

By $(PS)_c$, the condition $(PS)_c$ with $c$ defined by (1.38), holds.

Let $\eta : [0,1] \times X \to X$ be a $f$-decreasing homotopy according to Corollary 1.7 and $\gamma_1 = \eta(1, \gamma)$. Then 0 and $e$ belong to $\{x : |f(x) - c| \geq 2\varepsilon\}$ because $f(0) = 0$, $f(e) \leq 0$ and $c > 2\varepsilon$. By Corollary 1.7, (1) it follows that

$$\gamma_1(0) = \eta(1, \gamma(0)) = \eta(1, 0) = 0,$$

$$\gamma_1(1) = \eta(1, \gamma(1)) = \eta(1, e) = e,$$

which means that $\gamma_1 \in \Gamma$. By Corollary 1.7, (2) and (1.41) we obtain

$$\max_{t \in [0,1]} f(\gamma_1(t)) \leq c - \varepsilon,$$

which is a contradiction to the definition of $c$. Therefore $K_c \neq \emptyset$. End Proof

Deformation approach is used in various generalizations of Theorem 1.18. We refer to Rabinowitz [Ra2], Willem [Wil1], Schechter [Sch1].

Let us consider a characterization of a critical value $b$ as

$$b = \sup_{N \in \mathcal{N}} \inf_{x \in \partial N} f(x), \quad (1.42)$$

where

$$\mathcal{N} = \{N \subset X, N \text{bounded and open}, 0 \in N, e \notin \bar{N}\}.$$

The following result is an extension of Rabinowitz [Ra2], Theorem 3.2, that uses $(PS)_\psi$ condition (see [Sch1]).

**Theorem 1.19.** Let $f \in C^1(X, \mathbb{R})$, $e \neq 0$ and $N_0 \in \mathcal{N}$ be such that

(i) $f(0) \leq 0$ and $f(x) \geq a > 0$ if $x \in \partial N_0$,

(ii) $f(e) \leq 0$.

Let $\psi \in \Psi$ and suppose that $f$ satisfies $(PS)_{b,\psi}$ condition, where $b$ is defined by (1.42). Then $b$ is a critical value.

**Proof.** By (j) it follows that $b \geq a > 0$. Suppose by contradiction that $K_b = \emptyset$ and take $0 < \varepsilon < \frac{a}{2}$. Let $N_1 \in \mathcal{N}$ be such that
\[ \inf_{x \in \partial N_1} f(x) \geq b - \varepsilon \]

and \( M > 0 \) be such that \( N_0 \subset B_M \).

From Theorem 1.15, there exists a \( f \)-increasing homotopy

\[ \eta : [0, 1] \times X \to X, \]

such that

\[ \eta(1, f_{b-\varepsilon} \cap N_0) \subset f_{b+\varepsilon}, \]

because \( N_0 \subset B_M \) and \( b > 2\varepsilon \). Since \( \eta(1, \cdot) : X \to X \) is a homeomorphism, \( N_2 = \eta(1, N_1) \) is open and \( \partial N_2 = \eta(1, \partial N_1) \). Moreover \( 0 \in N_2 \) and \( e \notin N_2 \).

Indeed, since \( b > 2\varepsilon \) we have

\[ f(0) \leq 0 < b - 2\varepsilon, \quad f(e) \leq 0 < b - 2\varepsilon. \]

Then \( 0 \) and \( e \) belong to the set \( \{ x : |f(x) - b| \geq 2\varepsilon \} \) and, by Theorem 1.15 (1),

\[ 0 = \eta(1, 0) \in N_2, \quad e = \eta(1, e) \notin N_2. \]

So, \( N_2 \in \mathcal{N} \). And since \( f(\eta(1, x)) \geq b + \varepsilon \) if \( f(x) \geq b - \varepsilon \), and \( \partial N_1 \subset f_{b-\varepsilon} \)

we have

\[ \inf_{x \in \partial N_2} f(x) = \inf_{x \in \partial N_1} f(\eta(1, x)) \geq b + \varepsilon, \]

which is a contradiction to the definition of \( b \). Therefore \( K_b \neq \emptyset \). End Proof.

We extend Theorem 1.19 assuming nonstrict inequality in (j).

**Theorem 1.20.** Let \( f \in C^1(X, \mathbb{R}) \), \( 0 \) be a local minimum of \( f \) and there exist \( e \neq 0 \) such that \( 0 = f(0) \geq f(e) \). Let \( \psi \in \Psi \) and suppose that \( f \) satisfies \((PS)_\psi\) condition. Then there exists a critical point \( y \), different from \( 0 \) and \( e \).

**Proof.** Let \( \varepsilon \) be such that \( 0 < \varepsilon < ||e|| \) and \( f(x) \geq f(0) \) if \( ||x|| \leq \varepsilon \). We have the following alternative

(i) \( \exists \rho \in (0, \varepsilon) : c = \inf \{ f(x) : ||x|| = \rho \} > 0 \)

or

(ii) \( \forall \rho \in (0, \varepsilon) : \inf \{ f(x) : ||x|| = \rho \} = 0 \).

If (i) holds, since \( B_\rho(0) \in \mathcal{N} \) we have

\[ b = \sup_{N \in \mathcal{N}} \inf_{x \in \partial N} f(x) \geq c = \inf_{x \in S_\rho(0)} f(x) > 0. \]
The assertion follows by Theorem 1.19.
Let (ii) holds and take \( r \) and \( R \) such that \( 0 < r < R < \varepsilon \). Let \( (x_j)_j \) be a minimizing sequence of \( f \) on \( S_\rho \) where \( \rho \) is such that \( r < \rho < R \).
Define
\[
\bar{f}(x) = \begin{cases} 
  f\left(R \frac{x}{||x||}\right), & \text{if } ||x|| \geq R, \\
  f(x), & \text{if } ||x|| \leq R.
\end{cases}
\]
By Corollary 1.3, applied to \( \bar{f}(x) \), there exists \( y_j \in X \) such that
\[
\bar{f}(y_j) \leq \bar{f}(x_j),
\]
\[
||x_j - y_j|| \leq \frac{1}{\sqrt{j}},
\]
and
\[
\bar{f}(y_j) \leq \bar{f}(x) + \frac{1}{\sqrt{j}} ||x - y_j||, \quad \forall v \in X. \tag{1.43}
\]
For sufficiently large \( j \) the point \( y_j \) belongs to the interior of
\[
V = \{ x : r \leq ||x|| \leq R \}
\]
and \( \bar{f}(x_j) = f(x_j), \bar{f}(y_j) = f(y_j) \). If we take \( x = y_j + tv \) where \( ||v|| = 1 \), and pass to the limit in (1.43) as \( t \to 0 \) we obtain that
\[
||f'(y_j)|| \leq \frac{1}{\sqrt{j}}. \tag{1.44}
\]
Since \( \psi \in \Psi \) by \( r \leq ||y_j|| \leq R \) and (1.44) it follows
\[
\psi(||y_j||)||f'(y_j)|| \to 0.
\]
By \((PS)_\psi\) condition there exists a critical point \( y \) such that \( ||y|| = \rho \) for every \( \rho \) such that \( r < \rho < R \). End Proof

As a corollary of Theorem 1.20 we obtain the following “three critical point theorem” (which we refer as TCPT).

**Corollary 1.8.** Let \( f \in C^1(X, \mathbb{R}) \), \( \psi \in \Psi \) and suppose that \( f \) satisfies \((PS)_\psi\) condition. Suppose that \( f \) has two local minima. Then \( f \) has at least one more critical point.

Similar three critical points theorems with \((PS)\) condition were proved by Mawhin & Willem [MW1], Figueredo & Sollimini [FS], Pucci & Serrin [PS1].
In the variants of mountain-pass theorems considered above, we use deformation theorems proved in the previous section. Another approach to mountain-pass theorems is based on the Ekeland variational principle. We use it partially in the proof of Theorem 1.20. A general result in this direction is one of Aubin & Ekeland [AE]. To formulate their result we need another form of (PS) condition.

**Definition 1.10** (Aubin & Ekeland, 1984). The $C^1$-functional $f : X \to \mathbb{R}$ satisfies (WPS) condition on $\Omega \subset X$ if for every sequence $(x_j)$ in $\Omega$ such that

1. $|f(x_j)| \leq M$,
2. $f'(x_j) \neq 0$ for every $j \in \mathbb{N}$ and $\lim_j \|f'(x_j)\| = 0$,

there exists $\bar{x} \in X$ such that

$$\lim \inf_j f(x_j) \leq f(\bar{x}) \leq \lim \sup_j f(x_j), \quad f'(\bar{x}) = 0.$$

Relations between (PS) and (WPS) condition are given by a proposition proved in Aubin & Ekeland [AE].

**Proposition 1.1.** If $f : X \to \mathbb{R}$ satisfies (PS) condition on $X$, then $f$ satisfies (WPS) condition on $X$. If $X$ is a reflexive space, $f$ is convex, lower semicontinuous and coercive, then $f$ satisfies (WPS) condition on $X$.

Now, we formulate a variant of mountain-pass theorem with (WPS) condition proved in Aubin & Ekeland [AE]. For completeness we give the proof based on the Ekeland variational principle.

**Theorem 1.21** (Aubin & Ekeland, 1984). Let $f \in C^1(X, \mathbb{R})$ satisfy the following assumptions:

1. there exists $\alpha > 0$ such that
   $$m(\alpha) = \inf \{f(x) : \|x\| = \alpha\} > f(0),$$

2. there exists $e \in X$ such that
   $$\|e\| > \alpha, \quad f(e) < m(\alpha),$$

3. $f$ satisfies (WPS) condition on $\{x \in X : f(x) \geq m(\alpha)\}$. Then there exists $\bar{x} \in X$ such that $f(\bar{x}) \geq m(\alpha)$ and $f'(\bar{x}) = 0$.

**Proof.** We denote by $\mathcal{C}$ the set of paths joining the points 0 and $e$ in $X$
\[ C = \{ c \in C([0, 1], X) : c(0) = 0, c(1) = e \}, \]

equipped with the distance
\[ d(c_1, c_2) = \max \{ \| c_1(t) - c_2(t) \| : 0 \leq t \leq 1 \}. \]

The space \( (C, d) \) is a complete metric space. Let \( F : C \to \mathbb{R} \) be the functional
\[ F(c) = \max \{ f(c(t)) : 0 \leq t \leq 1 \}. \] (1.45)

It is lower semicontinuous and
\[ F(c) \geq m(\alpha). \]

Indeed for every \( c \in C \) there exists \( t_\alpha \in [0, 1] \) such that \( \| c(t_\alpha) \| = \alpha \) and then,
\[ F(c) \geq f(c(t_\alpha)) \geq m(\alpha). \]

By Ekeland principle, for every \( \varepsilon > 0 \) there exists \( c_\varepsilon \in C \), such that
\[ F(c_\varepsilon) \leq \inf_{c \in C} F(c) + \varepsilon, \]
and
\[ F(c) \geq F(c_\varepsilon) - \varepsilon d(c, c_\varepsilon), \quad \forall c \in C. \] (1.46)

Let \( \gamma \in C([0, 1], X) \) be such that \( \gamma(0) = \gamma(1) = 0 \). By (1.46) it follows that for \( h \in \mathbb{R}, h \neq 0 \),
\[ F(c_\varepsilon + h\gamma) - F(c_\varepsilon) \geq -\varepsilon d(c_\varepsilon + h\gamma, c_\varepsilon), \]
that is,
\[ \frac{1}{|h|} (F(c_\varepsilon + h\gamma) - F(c_\varepsilon)) \geq -\varepsilon \max_t \| \gamma(t) \|. \] (1.47)

On the other hand,
\[ F(c_\varepsilon + h\gamma) - F(c_\varepsilon) = \max_t f(c_\varepsilon(t) + h\gamma(t)) - \max_t f(c_\varepsilon(t)) \]
\[ = \max_t f(c_\varepsilon) + h \langle f'(c_\varepsilon(t)) \gamma(t), o(h) \rangle + o(h) - \max_t f(c_\varepsilon). \]
Let $f (c_\varepsilon (t)) = p (t)$ and $\langle f' (c_\varepsilon (t)), \gamma (t) \rangle = q (t)$. We have that $p$ and $q$ belong to $C ([0, 1], X)$. Let us introduce the functional $\Phi : C ([0, 1]) \to \mathbb{R}$ defined as

$$\Phi (\varphi) := \max_{t \in [0, 1]} \varphi (t),$$

for $\varphi \in C ([0, 1])$. The functional $\Phi$ is convex and the subdifferential $\partial \Phi$ is given by

$$\partial \Phi (\varphi) = \left\{ \mu \geq 0 : \int d\mu = 1, \quad \text{supp} (\mu) \subset M (\varphi) \right\},$$

where $\mu$ belongs to the space of Radon measures on $[0, 1]$ (see Cohn [Co]) and $M (\varphi) = \{ t : \varphi (t) = \Phi (\varphi) \}$. By (1.47) we obtain

$$-\varepsilon \max_t \| \gamma (t) \| \leq \lim_{h \to 0} \frac{1}{|h|} \left( F (c_\varepsilon + h\gamma) - F (c_\varepsilon) \right)$$

$$= \lim_{h \to 0} \frac{1}{|h|} \left( \Phi (p + hq) - \Phi (p) \right)$$

$$= \max \left\{ \langle q, \mu \rangle : \mu \in \partial \Phi (p) \right\}$$

$$= \max \left\{ \int \langle f' (c_\varepsilon), \gamma \rangle d\mu : \mu \in \partial \Phi (p) \right\}. $$

Taking $\gamma \in C ([0, 1], X)$, such that $\| \gamma \| \leq 1$, $\gamma (0) = \gamma (1) = 0$, by (1.48) and minimax theorem from Aubin & Ekeland [AE], Theorem 6.2.7, we have

$$-\varepsilon \leq \inf_{\gamma} \max_{\mu} \left\{ \int \langle f' (c_\varepsilon), \gamma \rangle d\mu : \begin{array}{c} \mu \in \partial \Phi (p) \\ \| \gamma \| \leq 1 \\ \gamma (0) = \gamma (1) = 0 \end{array} \right\}$$

and, then,

$$-\varepsilon \leq \max_{\mu} \inf_{\gamma} \left\{ \int \langle f' (c_\varepsilon), \gamma \rangle d\mu : \begin{array}{c} \mu \in \partial \Phi (p) \\ \| \gamma \| \leq 1 \\ \gamma (0) = \gamma (1) = 0 \end{array} \right\}$$

$$= \max \left\{ - \int \| f' (c_\varepsilon) \| d\mu : \mu \in \partial \Phi (p) \right\}$$

$$= - \min \left\{ \| f' (c_\varepsilon) \| : t \in M (f \circ c_\varepsilon) \right\}.$$
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\[ f(c_\varepsilon(t_\varepsilon)) = \max_t f(c_\varepsilon(t)), \]

and

\[ \|f'(c_\varepsilon(t_\varepsilon))\| \leq \varepsilon. \]

Taking now \( \varepsilon = \frac{1}{j} \) and \( c_{1/j}(t_{1/j}) = x_j \) we have proved that

\[ m(\alpha) \leq f(x_j) \leq \inf f + \frac{1}{j}, \]

and

\[ \lim_j \|f'(x_j)\| = 0. \]

By \((WPS)\) condition there exists \( \bar{x} \), such that

\[ f(\bar{x}) \geq \liminf f(x_j) \geq m(\alpha), \quad f'(\bar{x}) = 0. \quad \text{End Proof} \]

The idea of the proof of last theorem has been modified by Ghoussob & Preiss [GP] in order to get information about the location of critical points. They introduce \((PS)_{G,c}\) condition around a set \( G \) at the level \( c \) as follows.

**Definition 1.11** (Ghoussob & Preiss, 1989). The differentiable functional \( f : X \to \mathbb{R} \) satisfies \((PS)_{G,c}\) condition around a set \( G \) at the level \( c \) if every sequence \( (x_j)_j \) in \( X \) such that

1. \( \lim_j d(x_j, G) = 0, \)
2. \( \lim_j f(x_j) = c, \)
3. \( \lim_j \|f'(x_j)\| = 0, \)

has a convergent subsequence.

The usual \((PS)\) condition corresponds to the case where \((PS)_{G,c}\) is verified for any \( G \subset X \) and any \( c \in \mathbb{R} \).

**Definition 1.12.** A closed subset \( G \) of a Banach space separates two points \( u \) and \( v \) in \( X \) if \( u \) and \( v \) belong to disjoint connected components of \( X \setminus G \).

The following mountain-pass theorem has been proved in Ghoussoub & Preiss [GP].
**Theorem 1.22** (Ghoussoub & Preiss, 1989). Let $f : X \to \mathbb{R}$ be a Gâteaux-differentiable functional such that $f' : X \to X^*$ is continuous from $X$ with norm topology to $X^*$ with weak* topology. Fix $e \neq 0$ and define

$$c = \inf_{\gamma \in \Gamma} \max_{t \in [0, 1]} f(\gamma(t)),$$

where

$$\Gamma = \{ \gamma \in C([0, 1], X) : \gamma(0) = 0, \gamma(1) = e \}.$$

Let $G$ be a closed subset such that $G \cap f_c$ separates 0 and $e$. Assume that $f$ satisfies $(PS)_{G,c}$ condition. Then there exists $\bar{x} \in G$ such that $f(\bar{x}) = c$ and $f'(\bar{x}) = 0$.

The mountain-pass theorem follows from Theorem 1.22 with $G = X$.

The proof of Theorem 1.22 follows the idea of the proof of Theorem 1.21 but, instead of the functional $F(\gamma) = \max_{0 \leq t \leq 1} f(\gamma(t))$, it is considered the perturbed functional

$$I(\gamma) = \max_{0 \leq t \leq 1} (f(\gamma(t)) + \psi(\gamma(t))),$$

where

$$\psi(x) = \max\{0, \varepsilon^2 - \varepsilon d(x, G \cap f_c)\}.$$

Further Ekeland [Ek2] generalizes Theorem 1.22 assuming a variant of $(PS)_{G,c}$ condition in the sense of Cerami.

**Definition 1.13.** The differentiable functional $f : X \to \mathbb{R}$, satisfies $(PSC)_{G,c}$ condition around a set $G$ at the level $c$ if every sequence $(x_j)_j$ in $X$ such that:

1. $\lim_j d(x_j, G) = 0$,
2. $\lim_j f(x_j) = c$,
3. $\lim_j (1 + ||x_j||)||f'(x_j)|| = 0$,

has a convergent subsequence.

**Theorem 1.23** (Ekeland, 1990). Let $f : X \to \mathbb{R}$ be a Gâteaux-differentiable functional, such that $f' : X \to X^*$ is continuous from $X$ with norm topology to $X^*$ with weak* topology. Fix $e \neq 0$ and define

$$c = \inf_{\gamma \in \Gamma} \max_{t \in [0, 1]} f(\gamma(t)),$$

where

$$\Gamma = \{ \gamma \in C([0, 1], X) : \gamma(0) = 0, \gamma(1) = e \}.$$
Let $G$ be a closed subset such that $G \cap f_c$ separates 0 and $c$. Assume $f$ satisfies $(PSC)_{G,c}$ condition. Then there exists $\bar{x} \in G$ such that $f(\bar{x}) = c$ and $f'(\bar{x}) = 0$.

Let us note also the variant of mountain-pass theorem due to Brezis & Nirenberg [BN] without Palais–Smale condition. Let $Q$ be a compact metric space, $Q_*$ be a nonempty closed subset of $Q$ and $p_*$ be a fixed continuous map on $Q$. Define

$$\mathcal{P} := \{ p \in C(Q, X) : p(t) = p(t) \text{ if } t \in Q_* \}$$

and

$$c := \inf_{p \in \mathcal{P}} \max_{t \in Q} f(p(t)).$$

**Theorem 1.24** (Brezis & Nirenberg, 1991). Let $f \in C^1(X, \mathbb{R})$. Assume that for every $p \in \mathcal{P}$, $\max_{t \in Q} f(p(t))$ is attained at some point in $Q \setminus Q_*$. Then there exists a sequence $(x_j)_j$ in $X$ such that

$$\lim_j f(x_j) = c \quad \text{and} \quad \lim_j \| f'(x_j) \| = 0.$$ 

In addition, if $f$ satisfies $(PS)_c$ condition, then $c$ is a critical value. Moreover, if $(p_j)_j$ is any sequence in $\mathcal{P}$ such that

$$c = \lim_j \max_{t \in Q} f(p_j(t)),$$

then there exists a sequence $(t_j)_j$ in $Q$ such that

$$\lim_j f(p_j(t_j)) = c \quad \text{and} \quad \lim_j \| f'(p_j(t_j)) \| = 0.$$ 

We obtain Theorems 1.22 and 1.24 from a general mountain-pass theorem for locally Lipschitz functionals in Chapter 5.

Finally, we note that the mountain-pass theorem is proved also in scales of Banach spaces by Struwe [St1], [St2]. His result based on Deformation Theorem 1.16 in scales of Banach spaces is as follows

**Theorem 1.25.** Let $f : E \to \mathbb{R}$ be a $C^1$-functional, $(X_n)_n$ be a scale of subspaces of $X$ such that $E_n \subset E_{n+1}$ and $\bigcup_{n=1}^{\infty} E_n$ is dense in $E$. Assume that there exist $e \neq 0$, $\rho > 0$ and $\alpha > 0$ such that
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(1) \( f(0) = 0, \ f(u) \geq \alpha \) for every \( u \) with \( ||u|| = \rho \),
(2) \( f(e) < \alpha \),
(3) \( f \) satisfies \( (PS)_c^* \) condition with \( c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} f(\gamma(t)) \).

Then there exists \( v \in K_c \).

Proof. Assume that \( K_c \) is empty. By Theorem 1.17, for sufficiently small \( \varepsilon \) and sufficiently large \( n \), there exists a deformation \( \eta_n : [0,1] \times E_n \to E_n \) such that

\[
\eta_n(1, f^c + \varepsilon \cap E_n) \subset f^{c - \varepsilon}. \quad (1.49)
\]

Let \( \gamma \in \Gamma \) be such that

\[
c \leq \max_{t \in [0,1]} f(\gamma(t)) \leq c + \varepsilon,
\]

and \( \gamma_n = \eta_n(1, \gamma) \). By the properties of \( \eta_n \) and assumptions (1) and (2), it follows that \( \gamma_n \in \Gamma \). By (1.49), we have \( \max_{t \in [0,1]} f(\gamma_n(t)) \leq c - \varepsilon \) which contradicts to the definition of \( c \). End Proof
Bibliography


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